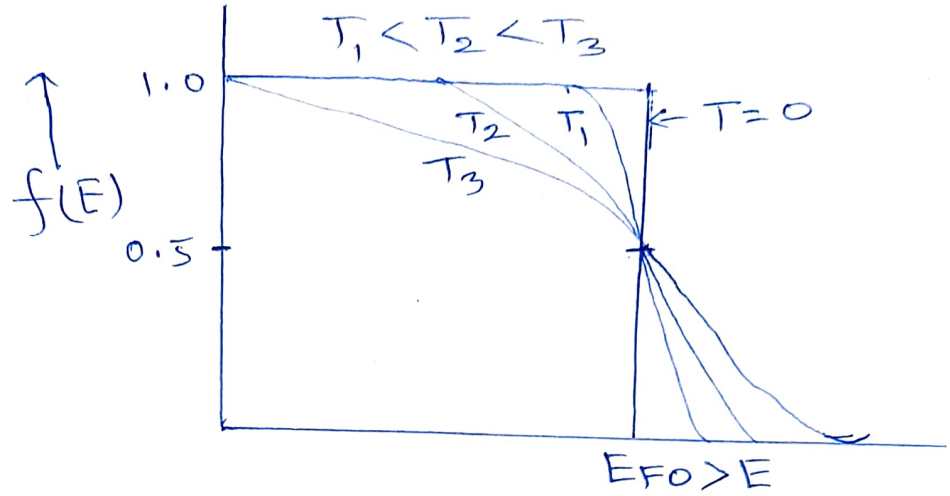
Where E_F is the Fermi energy and K_B is the Boltzmann constant. A plot of this function for several values of temperature T, including zero, is shown in Fig. below.



From figure it is observed that at T=0, the values of f(E) is unity for $E < E_{F0}$ and it is zero for $E > E_{F0}$, where E_{F0} is the Fermy energy at T=0. If $E = E_{F0}$, we have

$$f(E) = 0.5$$

At any temperature which does not differ very much from absolute zero, the value of f(E) is pratically unity for energy below E_{F0} so that $E_F - E \gg K_B T$. It is only in the vicinity of E_F minus a few $K_B T$ that f(E) begins to drop below the value at T=0 i.e. unity, indicating that all the states below E_F would not be filled and hence implying that all states above E_F will not be empty. It may therefore be concluded that if the electrons are heated up from absolute zero to a temperature T, each electron would not increase its energy (by $\sim K_B T$) as is expected classically, but those which are already within an energy range $K_B T$ below the Fermi level can do it. This is because the absorption of energy by an electron requires its excitation

into a higher energy state. For electrons lying deep into the energy there are no empty states within a few times k_BT into which they can be excited but for electrons lying below the Fermi level the empty states are available. Thus according to quantum theory, a very small fraction of electrons can absorb energy from an external heat source.

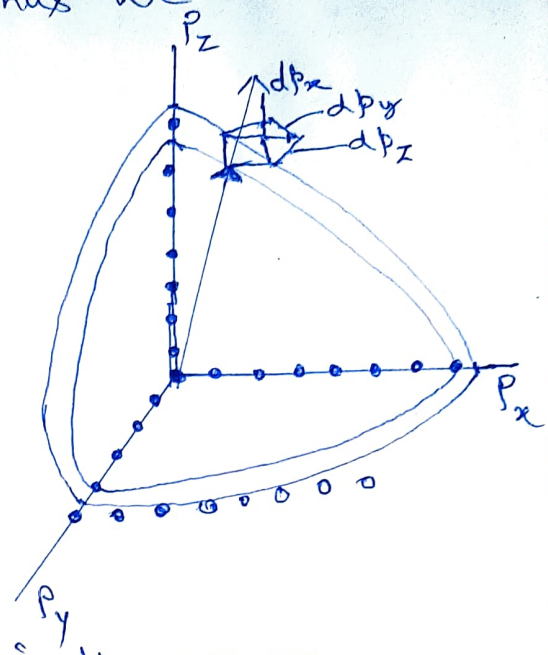
Density of available electron states D(E)

The total number of available electron states per unit energy range at energy value E is called density of available electron energy states and it is denoted by D(E). The energy corresponding to a set of values of k_x, k_y, k_z and m_s is called the energy state.

Let us plot the allowed values of the set of k_x, k_y, k_z, each set of k_x, k_y, k_z represents an allowed wave-function according to periodic boundary conditions. It is seen that the various points thus plotted in k-space form a simple orthorhombic lattice in the space with $\frac{2\pi}{L}, \frac{2\pi}{L}, \frac{2\pi}{L}$ as dimensions of the unit cell and $\left(\frac{2\pi}{L}\right)^3 = \frac{8\pi^3}{L^3} = \frac{8\pi^3}{V}$

where $V = L^3$, volume of the crystal as the volume of the k-space associated with a lattice point. Let us now calculate the quantity $D(k)dk$ which represents the number of k-values or the points in the wave-vector range dk and k. This is obtained by finding the volume of a spherical shell of radius k and thickness dk in the k-space (as this volume contains all the points having their wave vectors in the range dk at k) and dividing this

volume ($=4\pi k^2 dk$) by the volume associated with a single lattice point. Thus we have



$$D(k)dk = \frac{4\pi k^2 dk}{8\pi^3/V} = \frac{k^2 V}{2\pi^2} dk \quad \text{—(1)}$$

But $E = \frac{\hbar^2}{2m} k^2$

or $dE = \frac{\hbar^2}{2m} 2k\,dk$

or, $dk = \left(\frac{2m}{\hbar^2}\right)\frac{dE}{2k}$

$$\therefore D(k)dk = \frac{k^2 V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)\frac{dE}{2k}$$

The number of energy states in the range E and E + dE is given by

$$D(E)dE = \cancel{2}\cdot\frac{k^{\cancel{2}} V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)\frac{dE}{\cancel{2}k} \quad \text{—(2)}$$

$k^2 = \left(\frac{2m}{\hbar^2}\right)E$

$$= \frac{kV}{2\pi^2}\cdot\left(\frac{2m}{\hbar^2}\right)dE$$

$$= \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\cdot\frac{E^{\frac{1}{2}}V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{\frac{1}{2}}\,dE$$

$$\boxed{\therefore D(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{\frac{1}{2}}} \quad \text{—(3)}$$

Since each energy state represented by a set of k_x, k_y, k_z can accommodate two electrons of opposite spins, there is a multiplication by two in eqn. (2).

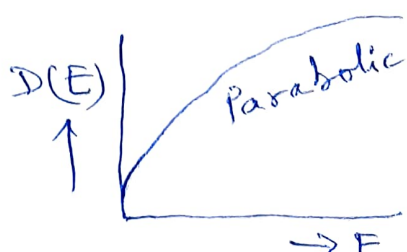
The expression for D(E) is generally written as

$$D(E) = CE^{\frac{1}{2}} \quad \text{——— (4)}$$

Where $C = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$

$\therefore$ Available energy per unit volume is

$$D(E) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}$$



The density of filled energy states N(E) :-

The density of filled energy states at energy E is the number of filled energy states per unit energy range at energy value E. It is denoted by N(E). The density of filled electron states N(E) at any temperature is given by

$$N(E) = D(E) f(E) \quad \text{——} \quad (5)$$

where

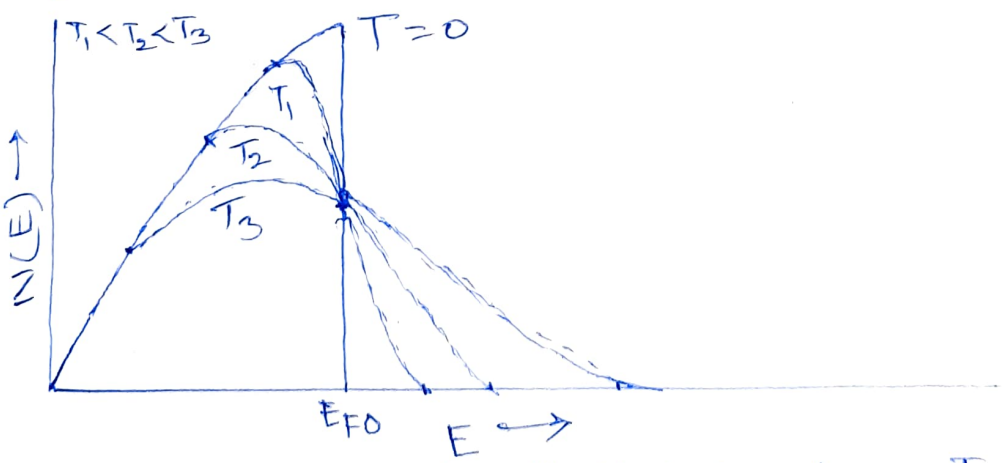
$$f(E) = \frac{1}{\exp[(E-E_F)/k_BT]+1}$$

Accordingly the number of filled energy states at any temperature T the energy range E and E+dE is given by

$$N(E)\,dE = D(E) f(E)\,dE \quad \text{——} \quad (6)$$

This also represents the number of electrons at temperature T in the energy range E and E+dE.

The plot of N(E) versus E is given in the following figure. We find that at $T_0 = 0$, all energy states below E_{F0} are



completely filled and all states above E_{F0} are completely empty. But as the temperature rises, the abrupt cut-off at E_{F0} becomes less and less pronounced and curve tails off

smoothly to zero. At temperatures such that $k_B T << E_F$, however, it is not very much deviated from that at $T = 0$.

Parameters of free electron gas at absolute zero

Fermi energy E_{F0}

If N be the total number of electrons (N being given) then.

$$N = \int_0^\infty N(E)\, dE$$

$$= \int_0^\infty D(E) F(E)\, dE$$

But at $T = 0$

$$f(E) = 1 \quad \text{for } E < E_{F0}$$

$$= 0 \quad \text{for } E > E_{F0}$$

We have

$$N = \int_0^{E_{F0}} (cE^{\frac{1}{2}}) \cdot 1 \cdot dE + \int_{E_{F0}}^\infty (cE^{\frac{1}{2}}) \cdot 0 \cdot dE$$

$$= \frac{2}{3} c E_{F0}^{3/2} = \frac{V}{3\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E_{F0}^{3/2}$$

Thus, $$E_{F0} = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$= \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3} \qquad n = \frac{N}{V}$$

The fermi energy may thus be calculated simply by knowing the electron concentration n.

— x —