

Forced Vibrations

Let us consider vibrations of a system on which an external force is acting. These are called forced vibrations. We assume that vibrations are small, so the external force is weak. ~~Forced vibrations~~

Let $U_e(x, t)$ be the potential energy due to the external force in addition to the potential energy $\frac{1}{2} K x^2$. Expanding $U_e(x, t)$ in power of x , we have

$$U_e(x, t) = U_e(0, t) + \left[\frac{\partial U_e}{\partial x} \right]_{x=0} + \dots \quad (1)$$

The first term is a function of time t only and so it may be omitted from the Lagrangian.

$$- \left[\frac{\partial U_e}{\partial x} \right]_{x=0} = F(t)$$

is the external force acting on the system in the equilibrium position. Then the potential energy due to this external force is

$$U_e(x, t) = -x F(t)$$

Hence the total potential energy of the system is

$$U(x, t) = \frac{1}{2} K x^2 - x F(t) \quad (2)$$

and the Lagrangian of the system is

$$L = \frac{1}{2} m \dot{x}^2 - U(x, t)$$

$$= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2 + x F(t) \quad (3)$$

The Lagrange equation of motion of the system is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad (4)$$

Substituting (3) in (4), we get the corresponding equation of motion

$$m \ddot{x} + K x - F(t) = 0$$

Thus

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m} \quad (5)$$

where

$$\omega = \sqrt{\frac{K}{m}}$$

is the frequency of the free oscillation.

Equation (5) is an inhomogeneous linear differential equation with constant co-efficient. The general solution of this equation is

$$x = x_0 + x_1, \quad \text{--- (6)}$$

Where x_0 is the general solution of the free oscillation and x_1 is a particular integral of the inhomogeneous equation.

Substituting (6) in (5), we have.

$$\ddot{x} + \omega^2 x_0 = 0 \quad \text{--- (7)}$$

and
$$\ddot{x}_1 + \omega^2 x_1 = \frac{F(t)}{m} \quad \text{--- (8)}$$

Eqⁿ (7) is the equation of motion of free oscillation.

Its solution is given by

$$x_0 = A \cos(\omega t + \delta) \quad \text{--- (9)}$$

Let us consider a case, which is of special interest, where the external force is a simple periodic function of time of frequency p .

$$F(t) = F_0 \cos(pt + \gamma) \quad \text{--- (10)}$$

Then eqⁿ (8) becomes

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= \frac{F_0}{m} \cos(pt + \gamma) \\ &= f_0 \cos(pt + \gamma) \end{aligned} \quad \text{--- (11)}$$

where $f_0 = F_0/m$

Let the general solution of (11) is

$$x_1 = B \cos(pt + \gamma) \quad \text{--- (12)}$$

Substituting this value in (11), we get

$$-B p^2 \cos(pt + \gamma) + \omega^2 B \cos(pt + \gamma) = f_0 \cos(pt + \gamma)$$

$$\text{or } B(\omega^2 - p^2) = f_0$$

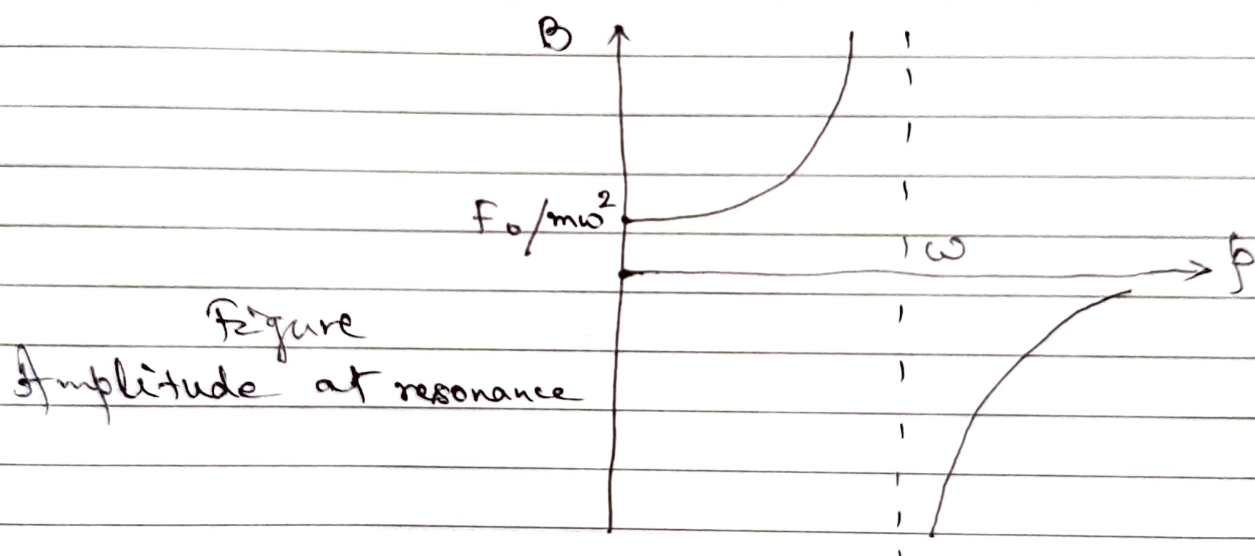
$$\text{or } B = \frac{f_0}{\omega^2 - p^2}$$

Thus we obtain
$$x_1 = \frac{f_0}{\omega^2 - p^2} \cos(pt + \gamma) \quad \text{--- (13)}$$

Then the general solution of (5) is

$$x = A \cos(\omega t + \delta) + \left(\frac{f_0}{\omega^2 - p^2} \right) \cos(pt + \gamma) \quad \text{--- (14)}$$

Where A and δ are arbitrary constants, which are found from the initial conditions. This shows that a system under the action of a periodic force executes a motion, which is resultant of two oscillations of frequencies ω and p ; ω is the natural frequency and p is the frequency of the force. The amplitude becomes large, when p is closed to ω . It is infinite at $p = \omega$ as shown in Fig. This is known as resonance.



The resonance occurs, when $p = \omega$, where solution (14) is not valid. In order to find the general solution of the equation of motion in this case, Eqn (14) can be written as

$$x = a \cos(\omega t + \delta) + \left(\frac{f_0}{(\omega^2 - p^2)} \right) [\cos(pt + \gamma) - \cos(\frac{\omega t}{2})]$$

$$= a \cos(\omega t + \delta) + \left(\frac{f_0 t}{2\omega} \right) \sin(\omega t + \delta) \quad (16)$$

Thus the amplitude of oscillations in resonance increase linearly with time.

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