M.Sc (Sem-II)

TOPIC:- Electromagnetic Field Due to Constant Motion of Point Charge:→

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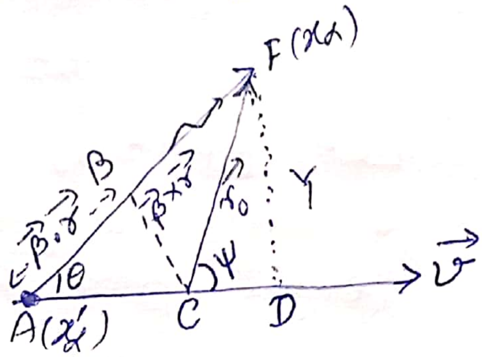
Let us consider a charge 'e' moving with uniform velocity $\vec{v}$ along +ve x-direction as shown in fig (1). Let $A(x'_\lambda)$ and $F(x_\lambda)$ be the respective source and field points such that

$$\vec{r} = x_\lambda - x'_\lambda.$$



Fig(1)

If a signal is emitted with speed c in all direction ~~then it will reach~~ from A, then it will reach at field point F in time r/c. During this time, the charge moves from A to C. Therefore points A and C are referred as retarded and present position of charge with respect to F. And retardation in distance r i.e BF is known as "Retarded distance S. Obviously BF = S.

From fig (1), If angle between $\vec{v}$ and $\vec{r}$ at A is θ, then

$$AB = AC\cos\theta$$
$$= \frac{vr}{c}\cos\theta \qquad (\because AC = \text{velocity} \times \text{time})$$
$$= \beta r\cos\theta$$
$$\therefore \vec{AB} = \vec{\beta}\cdot\vec{r} \qquad \text{where } \beta = v/c.$$

And
$$BC = AC\sin\theta$$
$$= \beta r\sin\theta$$
$$= \vec{\beta}\times\vec{r}$$

Hence the retarded distance in vector form is given as

$$\vec{S} = \vec{AF} - \vec{AB} = \vec{r} - \vec{\beta}\cdot\vec{r} \longrightarrow (1)$$

∴ whose magnitude is obtained from ΔCBF. In right angled ΔCBF, we have

$$BF^2 = (CF)^2 - (CB)^2$$
$$= r_0^2 - |\vec{\beta} \times \vec{r}|^2 \longrightarrow (2)$$

Let $CF (= r_0)$ be inclined at an angle ψ with $\vec{v}$ then the perpendicular length from F on x-axis is given by

$$Y = r_0 \sin\psi \quad \text{from } \Delta CDF$$

and from ΔADF,

$$Y = r \sin\theta$$

$$\therefore \quad r_0 \sin\psi = r \sin\theta$$

$$\text{or,} \quad \beta r_0 \sin\psi = \beta r \sin\theta$$

$$\text{or,} \quad \vec{\beta} \times \vec{r_0} = \vec{\beta} \times \vec{r} \longrightarrow (3)$$

from eq^n (2) and (3), we have

$$BF^2 = r_0^2 - |\vec{\beta} \times \vec{r_0}|^2$$
$$= r_0^2 - \beta^2 r_0^2 \sin^2\psi$$

$$\therefore S = BF = r_0 \sqrt{1 - \beta^2 \sin^2\psi} \longrightarrow (4)$$

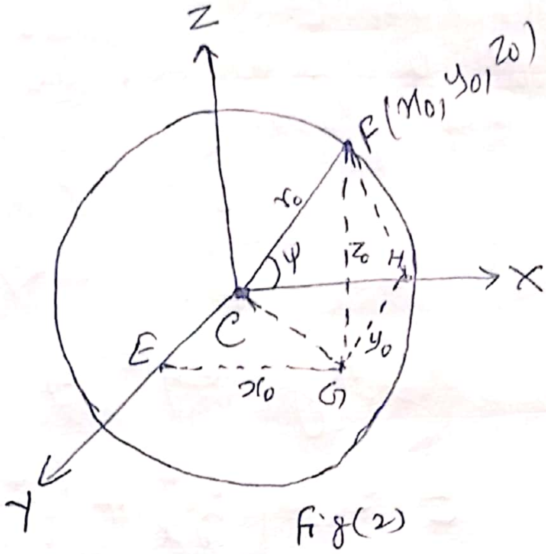
∴ According to Lienard and Wiechert, the value of scalar & vector potentials are given by

$$\phi = \frac{e}{4\pi\epsilon_0} \frac{1}{|\vec{S}|} = \frac{e}{4\pi\epsilon_0} \left[\frac{1}{r_0 \sqrt{1 - \beta^2 \sin^2\psi}} \right] \rightarrow (5)$$

$$\text{and} \quad A = \frac{\mu_0 e}{4\pi} \left[\frac{v}{|\vec{S}|} \right] = \frac{\mu_0 e c}{4\pi r_0} \left[\frac{\vec{\beta}}{(1 - \beta^2 \sin^2\psi)^{1/2}} \right] \rightarrow (6)$$

$$(\because \beta = v/c)$$

As signal is emitted spherically, we use the spherical polar co-ordinate here. If Cartesian co-ordinates of point F w.r.to C be (x_0, y_0, z_0), then from fig ②, we have



Fig(2)

$$r_0^2 = (CG)^2 + (FG)^2$$
$$= x_0^2 + y_0^2 + z_0^2$$

and $\sin\psi = \frac{FH}{CF} = \frac{\sqrt{y_0^2+z_0^2}}{r_0}$

$$\therefore \sin^2\psi = \frac{y_0^2+z_0^2}{r_0^2}$$

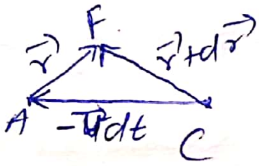
$$\therefore r_0\sqrt{1-\beta^2\sin^2\psi} = r_0\sqrt{1-\frac{\beta^2(y_0^2+z_0^2)}{r_0^2}}$$
$$= \sqrt{r_0^2-\beta^2(y_0^2+z_0^2)}$$
$$= \sqrt{(x_0^2+y_0^2+z_0^2)-\beta^2(y_0^2+z_0^2)}$$

$\therefore$ Eq^n ⑤ and ⑥ become in terms of Cartesian co-ordinate as

$$\phi = \frac{e}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x_0^2+y_0^2+z_0^2)-\beta^2(y_0^2+z_0^2)}}\right] \rightarrow ⑦$$

and $$A = \frac{\mu_0 ec}{4\pi}\left[\frac{\vec{\beta}}{\sqrt{(x_0^2+y_0^2+z_0^2)-\beta^2(y_0^2+z_0^2)}}\right] \rightarrow ⑧$$

Suppose position of charge change from $\vec{r}$ to $\vec{r}+d\vec{r}$ in time dt, then from fig ③, we have



Fig(3)

$$-\vec{v}dt + \vec{r} = \vec{r}+d\vec{r}$$
$$\therefore d\vec{r} = -\vec{v}dt$$
$$\therefore \frac{d\vec{r}}{dt} = -\vec{v} = -\vec{v}\frac{d\vec{r}}{d\vec{r}} \Rightarrow \frac{\partial}{\partial t} = -\vec{v}\frac{\partial}{\partial r} \rightarrow ⑨$$

We have to calculate the electromagnetic field $\vec{E}$ & $\vec{B}$ due to constant motion of a charge, so we use the field equations

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{---(a)}$$

$$\& \quad \vec{E} = -\nabla\phi - \frac{\partial A}{\partial t} \quad \text{---(b)} \qquad (10)$$

using eq^n (a) and (10 b), we have

$$\vec{E} = -\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\right) + \vec{v}\frac{\partial \vec{A}}{\partial r}$$

Its x-component is

$$E_x = -\frac{\partial \phi}{\partial x} + \vec{v}\frac{\partial \vec{A}}{\partial x}$$

$$= -\frac{\partial}{\partial x}\left[\frac{e}{4\pi\epsilon_0}\cdot\frac{1}{S}\right] + \vec{v}\frac{\partial}{\partial x}\left[\frac{e\mu_0\epsilon_0\vec{v}}{4\pi\epsilon_0}\cdot\frac{1}{S}\right]$$

$$= -\frac{e}{4\pi\epsilon_0}\left(1-\frac{v^2}{c^2}\right)\frac{\partial}{\partial x}\left(\frac{1}{S}\right)$$

$$= -\frac{e}{4\pi\epsilon_0}(1-\beta^2)\cdot\frac{\partial}{\partial x}\left(\frac{1}{S}\right) \quad (\because \beta = v/c)$$

$$= -\frac{e}{4\pi\epsilon_0}(1-\beta^2)\cdot\frac{\partial}{\partial x}\left[(x_0^2+y_0^2+z_0^2)-\beta^2(y_0^2+z_0^2)\right]^{-1/2}$$

$$= -\frac{e}{4\pi\epsilon_0}(1-\beta^2)\left[-\frac{1}{2}\{(x_0^2+y_0^2+z_0^2)-\beta^2(y_0^2+z_0^2)\}^{-3/2} 2x_0\right]$$

$$= \frac{e}{4\pi\epsilon_0}(1-\beta^2)\left[\frac{x_0}{S^3}\right] \longrightarrow (11)$$

Similarly Y and Z components are written as

$$E_y = \frac{e}{4\pi\epsilon_0}(1-\beta^2)\left[\frac{y_0}{S^3}\right] \longrightarrow (12)$$

$$\& \quad E_z = \frac{e}{4\pi\epsilon_0}(1-\beta^2)\left[\frac{z_0}{S^3}\right] \longrightarrow (13)$$

∴ Resultant electric field is given by

$$\vec{E} = \vec{i}E_x + \vec{j}E_y + \vec{k}E_z = \frac{e}{4\pi\epsilon_0}(1-\beta^2)\frac{(\vec{i}x_0 + \vec{j}y_0 + \vec{k}z_0)}{S^3}$$

$$\therefore \vec{E} = \frac{e}{4\pi\epsilon_0} \cdot \frac{(1-\beta^2)\,\vec{r}_0}{(r_0^2 - r_0^2\beta^2\sin^2\psi)^{3/2}}$$

$$= \frac{e}{4\pi\epsilon_0} \cdot \frac{\vec{r}_0}{r_0^3} \frac{(1-\beta^2)}{(1-\beta^2\sin^2\psi)^{3/2}} \longrightarrow (14)$$

Magnetic field $\vec{B}$:- From equation (10a), we have

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & 0 & 0 \end{vmatrix} = \vec{i}(0) + \vec{j}\left(\frac{\partial A}{\partial z}\right) + \vec{k}\left(-\frac{\partial A}{\partial y}\right)$$

$$\Rightarrow, B_x = 0, \quad B_y = \frac{\partial A}{\partial z} \quad \& \quad B_z = -\frac{\partial A}{\partial y}.$$

Here, $B_y = \frac{\partial A}{\partial z} = \frac{\partial}{\partial z}\left[\frac{\mu_0 e v \epsilon_0}{4\pi\epsilon_0}\left(\frac{1}{s}\right)\right] = \frac{ev}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial z}\left(\frac{1}{s}\right)$

$$= \frac{ev}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial z}\left[\{x_0^2+y_0^2+z_0^2\} - \beta^2(y_0^2+z_0^2)\}\right]^{-1/2}$$

$$= \frac{ev}{4\pi\epsilon_0 c^2}\left(-\frac{1}{2}\right)\frac{(2z_0 - 2z_0\beta^2)}{(r_0^2 - r_0^2\beta^2\sin^2\psi)^{3/2}}$$

$$= -\frac{ev}{4\pi\epsilon_0 c^2}\frac{(1-\beta^2)z_0}{r_0^3(1-\beta^2\sin^2\psi)^{3/2}} \longrightarrow (15)$$

Similarly, $B_z = -\frac{\partial A}{\partial y} = \frac{ev}{4\pi\epsilon_0 c^2}\frac{(1-\beta^2)y_0}{r_0^3(1-\beta^2\sin^2\psi)^{3/2}} \longrightarrow (16)$

Now, as $\vec{\beta} \times \vec{r}_0 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \beta & 0 & 0 \\ x_0 & y_0 & z_0 \end{vmatrix} = \vec{i}(0) + \vec{j}(-\beta z_0) + \vec{k}(\beta y_0)$

$\therefore (\vec{\beta} \times \vec{r}_0)_x = 0$; $(\vec{\beta} \times \vec{r}_0)_y = -\beta z_0$ & $(\vec{\beta} \times \vec{r}_0)_z = \beta y_0$

Hence eqn (15) & (16) becomes

$$B_y = \frac{e(1-\beta^2)}{4\pi\epsilon_0 c} \cdot \frac{(-\beta z_0)}{r_0^3(1-\beta^2\sin^2\psi)^{3/2}}$$

$$\& \; B_z = \frac{e(1-\beta^2)}{4\pi\epsilon_0 c}\frac{(\beta y_0)}{r_0^3(1-\beta^2\sin^2\psi)^{3/2}}$$

∴ Magnetic field $\vec{B} = \hat{i} B_x + \hat{j} B_y + \hat{k} B_z$ will become

$$\vec{B} = \frac{e(1-\beta^2)}{4\pi\epsilon_0 c} \cdot \frac{(\vec{\beta} \times \vec{r_0})}{r_0^3 (1-\beta^2 \sin^2\psi)^{3/2}} \longrightarrow (17)$$

Eq^n (14) and (17) are the required expression ~~for~~ the electric and magnetic fields due to ~~moving~~ uniformly moving point charge. From ~~these~~ eq^{ns}, we have

$$\vec{B} = \frac{\vec{\beta} \times \vec{E}}{c} = \frac{\vec{v} \times \vec{E}}{c^2} \longrightarrow (18)$$

Thus the magnetic field $\vec{B}$ is at right angles to a plane containing $\vec{v}$ and $\vec{E}$.