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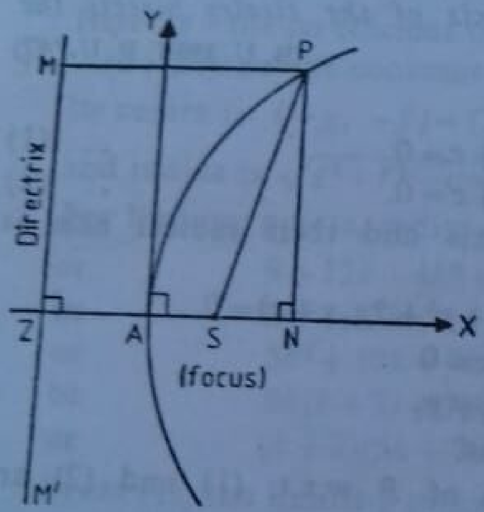
For T.D.C. Part I
Paper – 2
2 - dimensions geometry

1. *Define a parabola and obtain its equation in the standard form.*
(P. U. 1963; M. U. '70)

Solution. Definition. The parabola is the locus of a point which moves such that its distance from the fixed point, called the focus, is equal to its distance from the fixed straight line, called the directrix.



We shall now deduce the equation to a parabola.

Let S be the focus and ZM the directrix of the parabola.

From S draw $\perp$ SZ on the directrix.

Let A be the mid-point of SZ.

$\therefore$ $AS=AZ$, hence A lies on the parabola.

Let A be the origin and AX, AY the axes of co-ordinates.

Let P be any point (x, y) on the parabola.

Draw PM and PN perpendiculars on the directrix and AX respectively. Join S and P

Let $SZ=2a$, so that

$$AS=a \quad \text{and} \quad AZ=a.$$

Hence the co-ordinates of S are $(a, 0)$.

$$\therefore \quad SP^2=(x-a)^2+(y-0)^2. \qquad \text{.. (1)}$$

By definition of the parabola,

$$SP=PM=ZN=AN+AZ=x+a. \qquad \text{.. (2)}$$

From (1) and (2),

$$(x-a)^2+(y-0)^2=SP^2=(x+a)^2$$

or

$$y^2=(x+a)^2-(x-a)^2$$
$$=4ax.$$

Hence $y^2=4ax$ is the standard equation of the parabola.

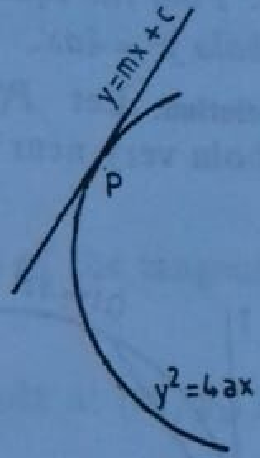
2. *Find the condition that the line $y=mx+c$ may touch the parabola $y^2=4ax$.*

(B. U. 1962)

Solution. The parabola is $y^2=4ax$ .. (1)

and the straight line is $y=mx+c$. .. (2)

The points of intersection of the line and the parabola are obtained by solving (1) and (2) simultaneously.



From (1) and (2), we have

$$(mx+c)^2=4ax$$

or $$m^2x^2+2x(mc-2a)+c^2=0. \quad .. (3)$$

If the given line touches the parabola, then the two values of x given by (3) must be equal.

The condition for which is $b^2=4ac$, formula

or $$4(mc-2a)^2=4m^2c^2$$

or $$m^2c^2+4a^2-4acm=m^2c^2$$

or $$4a^2=4acm$$

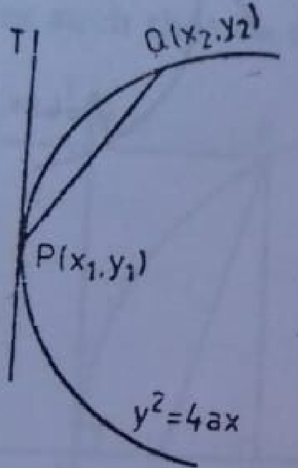
or $$c=\frac{a}{m}.$$

This is the required condition of tangency.

3. *Find the equation of the tangent at any point* (x_1, y_1) *of the parabola* $y^2=4ax$.

(M. U. 1967, '63; P. U. '56)

Solution. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the parabola very near to each other.



Since both the points are on the parabola, therefore

$$y_1^2=4ax_1 \quad \cdots (1)$$

and

$$y_2^2=4ax_2. \quad \cdots (2)$$

$$\therefore \quad y_2^2-y_1^2=4a(x_2-x_1)$$

or

$$\frac{y_2-y_1}{x_2-x_1}=\frac{4a}{y_2+y_1}. \quad \cdots (3)$$

The equation of the chord PQ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

or $y-y_1=\dfrac{4a}{y_2+y_1}(x-x_1)$, by virtue of (3)

or $y(y_1+y_2)-4ax-y_1y_2=0$ .. (4), by virtue of (1).

This is the equation of the chord joining

(x_1, y_1) and (x_2, y_2).

Now let the point Q coincide with the point P so that

$x_2 \to x_1$ and $y_2 \to y_1$

and PQ becomes the tangent at P.

Tangent at (x_1, y_1) is

$$y.2y_1-4ax-y_1^2=0$$

or $\quad 2yy_1-4ax-4ax_1=0$, by virtue of (1)

or

$$yy_1=2a(x+x_1). \quad \cdots (5)$$

This is the required equation of the tangent whose slope is $\dfrac{2a}{y_1}$.

4. *Find the equation of the normal to the parabola $y^2=4ax$ at the point (x_1, y_1).*

(Bh. U. 1969; B. U. '60; P. U. '51, 'Mith. U. '86)

Solution. The equation of the tangent at (x_1, y_1) to the parabola $y^2=4ax$ is

$$yy_1=2a(x+x_1)$$

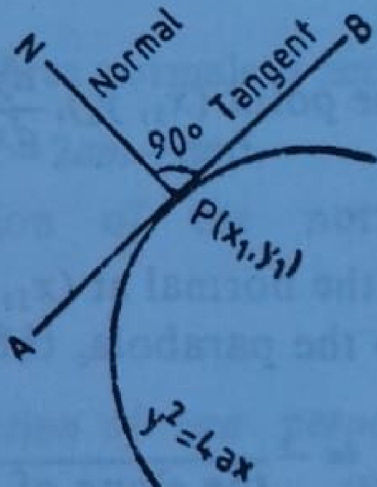
or

$$2ax-yy_1+2ax_1=0.$$

The slope of this tangent

$$=-\frac{\text{co-efficient of } x}{\text{co-efficient of } y}$$

$$=-\frac{2a}{-y_1}=\frac{2a}{y_1}.$$



Since the normal is perpendicular to the tangent, therefore the slope of the normal × the slope of the tangent $=-1$

or the slope of the normal $\times \frac{2a}{y_1} = -1$

or the slope of the normal $= -\frac{y_1}{2a}$,

Hence the equation of the normal at (x_1, y_1) to the parabola is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

5. *Show that three normals can, in general, be drawn to a parabola from any point and these normals are such that*

(i) *the sum of their slopes* $=0$ (Bh. U. 1965; M. U. '84; R. U. '68, '59)

(ii) *the sum of the ordinates of their feet* $=0$. (M. U. 1972)

Solution. (i) Let the equation of the parabola be

$$y^2=4ax. \quad \text{.. (1)}$$

Then the equation of the normal at $(am^2, -2am)$ is

$$y=mx-2am-am^3. \quad \text{.. (2)}$$

If the normal passes through a fixed point $Q\ (x_1, y_1)$, then

$$y_1=mx_1-2am-am^3$$

or

$$am^3+(2a-x_1)m+y_1=0. \quad \text{.. (3)}$$

This equation is a cubic equation in m; hence it gives three values of m, say m_1, m_2, m_3. Corresponding to each of these slopes, we get a normal passing through the point (x_1, y_1).

Hence from any point, three normals can in general be drawn to a parabola.

Since m_1, m_2, m_3 are the root of (3),

$$\therefore \quad m_1+m_2+m_3=-\frac{\text{the co-efficient of } m^2}{\text{the co-efficient of } m^3}=\frac{0}{a}=0.$$

Hence the sum of the slopes of the normals of the parabola $y^2=4ax$ drawn from any point is zero.

(ii) The sum of the ordinates of the feet of the three normals

$$=-2am_1-2am_2-2am_3$$
$$=-2a(m_1+m_2+m_3)$$
$$=-2a\times 0$$
$$=0.$$

Hence the sum of the ordinates of the three normals is zero.