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Study materials for T.D.C.
Part III, Paper-6, Group-A.

Directs Products of Groups

External Direct Products

The Cartesian Product of two groups G_1 & G_2 under the operation multiplication is defined as

$$G_1 \times G_2 = \{(a,b) : a \in G_1, b \in G_2\}$$

If we define the operation of multiplication between two elements (x_1, x_2) & (y_1, y_2) of $G_1 \times G_2$ as follows:

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2),$$

then $G_1 \times G_2$ is a group w.r.t. the above operation & this group is called the external direct product of G_1 & G_2.

Q. If G_1 & G_2 are two groups, then show that $G_1 \times G_2$ is a group under a suitable binary operation of multiplication.

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in G_1 \times G_2$ where $x_1, y_1, z_1 \in G_1$ & $x_2, y_2, z_2 \in G_2$

We define the operation of multiplication on $G_1 \times G_2$ as follows:

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2)$$

Now we verify the group postulates one by one

(i) Closure law:

Let $(x_1, x_2), (y_1, y_2) \in G_1 \times G_2$

Then $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2) \in G_1 \times G_2$, since $x_1 y_1 \in G_1$ & $x_2 y_2 \in G_2$

Hence closure law is satisfied

(ii) Associative law:

We have $(x_1, x_2)[(y_1, y_2)(z_1, z_2)]$

$$= (x_1, x_2)(y_1 z_1, y_2 z_2)$$
$$= (x_1 [y_1 z_1], x_2 [y_2 z_2])$$
$$= ([x_1 y_1] z_1, [x_2 y_2] z_2)$$
$$= (x_1 y_1, x_2 y_2)(z_1, z_2)$$
$$= [(x_1, x_2)(y_1, y_2)](z_1, z_2), \text{ where } (x_1, x_2), (y_1, y_2), (z_1, z_2) \in G_1 \times G_2$$

Hence associative law is satisfied

(iii) Existence of identity element.

Let e_1 & e_2 be the identity elements of G_1 & G_2 respectively. Then $(e_1, e_2) \in G_1 \times G_2$

Now, $(x_1, x_2)(e_1, e_2) = (x_1 e_1, x_2 e_2) = (x_1, x_2)$

Similarly, $(e_1, e_2)(x_1, x_2) = (e_1 x_1, e_2 x_2) = (x_1, x_2)$

$\therefore (x_1, x_2)(e_1, e_2) = (e_1, e_2)(x_1, x_2) = (x_1, x_2)$

Hence (e_1, e_2) is the identity element of $G_1 \times G_2$

(iv) Existence of inverse element:

Let $x_1^{-1} \in G_1$ & $x_2^{-1} \in G_2$ be the inverses of $x_1 \in G_1$ & $x_2 \in G_2$ respectively. Then $(x_1^{-1}, x_2^{-1}) \in G_1 \times G_2$

Now, $(x_1, x_2)(x_1^{-1}, x_2^{-1}) = (x_1 x_1^{-1}, x_2 x_2^{-1})$

$= (e_1, e_2)$

$= (x_1^{-1}, x_2^{-1})(x_1, x_2)$

Hence (x_1^{-1}, x_2^{-1}) is the inverse element of $(x_1, x_2) \in G_1 \times G_2$

Thus we see that all the group postulates are satisfied by $G_1 \times G_2$ under the operation defined above.

Hence $G_1 \times G_2$ is a group.

Q. If G_1 & G_2 are groups, then prove that the subsets $G_1 \times \{e_2\}$ & $\{e_1\} \times G_2$ of $G_1 \times G_2$ are normal subgroups of $G_1 \times G_2$ isomorphic to G_1 & G_2 respectively

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We have $G_1 \times \{e_2\} \subseteq G_1 \times G_2$

Let (x_1, e_2) & $(y_1, e_2) \in G_1 \times \{e_2\}$, where $x_1, y_1 \in G_1$ & $e_2 \in \{e_2\}$; e_2 being the identity of G_2.

Then $(x_1, e_2)(y_1, e_2)^{-1} = (x_1, e_2)(y_1^{-1}, e_2^{-1})$

$= (x_1 y_1^{-1}, e_2 e_2^{-1})$

$= (x_1 y_1^{-1}, e_2) \in G_1 \times \{e_2\}$

This shows that $G_1 \times \{e_2\}$ is a subgroup of $G_1 \times G_2$.

Now we shall show that $G_1 \times \{e_2\}$ is the normal subgroup of $G_1 \times G_2$

Let $(x_1, e_2) \in G_1 \times \{e_2\}$ and $(y_1, y_2) \in G_1 \times G_2$, where $x_1, y_1 \in G_1$ & $y_2 \in G_2$

Then $(y_1, y_2)(x_1, e_2)(y_1, y_2)^{-1} = (y_1, y_2)(x_1, e_2)(y_1^{-1}, y_2^{-1})$

$= (y_1 x_1 y_1^{-1}, y_2 e_2 y_2^{-1})$

$= (y_1 x_1 y_1^{-1}, y_2 y_2^{-1})$

$= (y_1 x_1 y_1^{-1}, e_2) \in G_1 \times \{e_2\}$

This shows that $G_1 \times \{e_2\}$ is the normal subgroup of $G_1 \times G_2$

Similarly we can show that $\{e_1\} \times G_2$ is the normal subgroup of $G_1 \times G_2$.

We shall now show that $G_1 \times \{e_2\}$ is isomorphic to G_1

Let us define a mapping $f : G_1 \to G_1 \times \{e_2\}$ such that

$f(x_1) = (x_1, e_2) \ \forall \ x_1 \in G_1$

clearly f is one-one onto mapping

Also if $x_1, y_1 \in G_1$, then $f(x_1 y_1) = (x_1 y_1, e_2)$

$= (x_1, e_2)(y_1, e_2)$

$= f(x_1) f(y_1)$

Thus f is an isomorphism & therefore $G_1 \cong G_1 \times \{e_2\}$

Similarly we can prove that $G_2 \cong \{e_1\} \times G_2$

Q. If G_1 & G_2 are two groups & e_1, e_2 are the identities of G_1 & G_2 respectively, then prove that

(i) $[G_1 \times \{e_2\}] \cap [\{e_1\} \times G_2] = \{(e_1, e_2)\}$

(ii) Every element of $G_1 \times \{e_2\}$ commutes with every element of $\{e_1\} \times G_2$

(iii) Every element of $G_1 \times G_2$ can be uniquely expressed as the product of an element of $G_1 \times \{e_2\}$ by an element of $\{e_1\} \times G_2$

(iv) $G_1 \times G_2 \cong G_2 \times G_1$

Proof:- (i) Let $(x_1, x_2) \in [G_1 \times \{e_2\}] \cap [\{e_1\} \times G_2]$

$\Rightarrow (x_1, x_2) \in G_1 \times \{e_2\}$ and $(x_1, x_2) \in \{e_1\} \times G_2$

$\Rightarrow x_1 \in G_1, x_2 \in \{e_2\}$ and $x_1 \in \{e_1\}, x_2 \in G_2$

$\Rightarrow x_1 \in G_1, x_2 = e_2$ and $x_1 = e_1, x_2 \in G_2$

$\Rightarrow (x_1, x_2) = (e_1, e_2)$

Similarly all other elements of $[G_1 \times \{e_2\}] \cap [\{e_1\} \times G_2]$ are equal to (e_1, e_2)

$\therefore [G_1 \times \{e_2\}] \cap [\{e_1\} \times G_2] = \{(e_1, e_2)\}$

(ii) Let $(x_1, e_2) \in G_1 \times \{e_2\}$ & $(e_1, x_2) \in \{e_1\} \times G_2$

$\implies x_1 \in G_1, \ x_2 \in G_2$

Now
$$\begin{aligned}(x_1, e_2)(e_1, x_2) &= (x_1 e_1, e_2 x_2) \\ &= (e_1 x_1, x_2 e_2) \\ &= (e_1, x_2)(x_1, e_2)\end{aligned}$$

Hence every element of $G_1 \times \{e_2\}$ commutes with every element of $\{e_1\} \times G_2$

(iii) Let $(x_1, x_2) \in G_1 \times G_2$

$\implies x_1 \in G_1, \ x_2 \in G_2$

Now
$$\begin{aligned}(x_1, x_2) &= (x_1 e_1, x_2 e_2) \\ &= (x_1 e_1, e_2 x_2) \\ &= (x_1, e_2)(e_1, x_2)\end{aligned}$$

i.e. every element of $G_1 \times G_2$ can be expressed as the product of an element of $G_1 \times \{e_2\}$ by an element of $\{e_1\} \times G_2$

We shall now show that this expression is unique.

If possible, let $(x_1, x_2) = (y_1, e_2)(e_1, y_2)$

$\implies (x_1, x_2) = (y_1 e_1, e_2 y_2)$

$\implies (x_1, x_2) = (y_1, y_2)$

$\implies x_1 = y_1, \ x_2 = y_2$

$\therefore (y_1, e_2)(e_1, y_2) = (x_1, e_2)(e_1, x_2)$

This shows that the above expression is unique.

(iv) Let us define a mapping $f: G_1 \times G_2 \to G_2 \times G_1$ such that

$f[(x_1, x_2)] = (x_2, x_1) \ \forall (x_1, x_2) \in G_1 \times G_2$

Clearly f is one-one onto mapping

Also f is homomorphism because if $(x_1, x_2), (y_1, y_2) \in G_1 \times G_2$,

then
$$\begin{aligned}f[(x_1, x_2)(y_1, y_2)] &= f(x_1 y_1, x_2 y_2) \\ &= (x_2 y_2, x_1 y_1) \\ &= (x_2, x_1)(y_2, y_1) \\ &= f[(x_1, x_2)]\, f[(y_1, y_2)]\end{aligned}$$

Thus f is an isomorphism & hence $G_1 \times G_2 \cong G_2 \times G_1$