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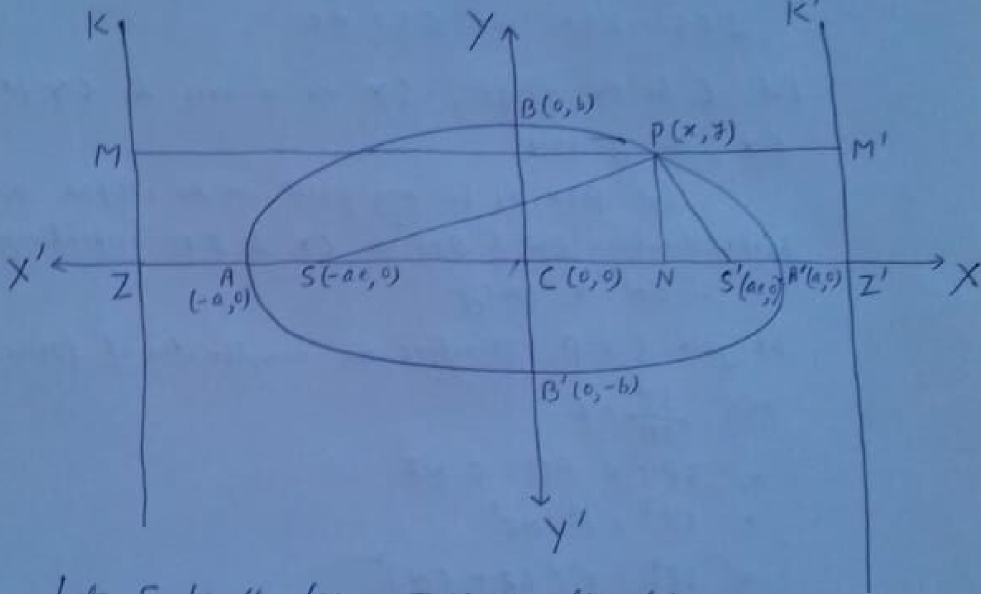
Study materials
for T.D.C. Part I
Paper - 2
2 - dimensions geometry

Q. Define an ellipse & Obtain its equation in standard form.

An ellipse is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point in the plane to its distance from a fixed straight line in the same plane is constant and less than unity.

The fixed point is called the focus, the fixed straight line is called the directrix and the constant ratio is called the eccentricity.

Equation of the ellipse:



Let S be the focus, ZM the directrix and e the eccentricity of the ellipse.

Let SZ be the perpendicular to the directrix ZM. Since $e<1$, so let us take a point A on SZ such that $\frac{SA}{AZ}=e$.

or, $SA = e.AZ$ —— (1)

Hence A must lie on the ellipse.

Since $e<1$, therefore we can also find a point A' on ZS produced such that $\frac{SA'}{A'Z}=e$

or, $SA' = e.A'Z$ —— (2)

Hence A' will also lie on the ellipse.

Let $AA' = 2a$ & let C be the middle point of AA'

$\therefore CA = a$ & $CA' = a$

Now, $CA - CS = AS = e.AZ$ [from (1)]
$= e(CZ - CA)$

or, $a - CS = e(CZ - a)$ —— (3)

Again, $CA' + CS = SA' = e.A'Z$ [from (2)]
$= e(CZ + CA')$

or, $a + CS = e(CZ + a)$ —— (4)

Adding (3) & (4), we get

$2a = 2e.CZ$ or, $a = e.CZ$

Subtracting (3) from (4), we get

$2CS = 2ae$ or, $CS = ae$

Let C be the origin, CX the x-axis & CY perpendicular to CX, the y-axis.

Let $P(x, y)$ be any point on the ellipse. We draw perpendiculars PN & PM to CX & ZM respectively.

$\therefore CN = x,\ PN = y$

We join S & P. Therefore the coordinates of focus S are $(-ae, 0)$

Now, $\frac{SP}{PM} = e$

or, $SP = e.PM = e.NZ$

or, $SP^2 = e^2.NZ^2$

or, $SP^2 = e^2(CZ + CN)^2$

or, $[x - (-ae)]^2 + (y - 0)^2 = e^2\left(\frac{a}{e} + x\right)^2$

or, $(x + ae)^2 + y^2 = e^2\left(\frac{a^2}{e^2} + 2\frac{a}{e}x + x^2\right)$

or, $x^2 + \cancel{2xae} + a^2e^2 + y^2 = a^2 + \cancel{2aex} + e^2x^2$

or, $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

Dividing both sides by $a^2(1-e^2)$, we get

$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

or, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1-e^2)$

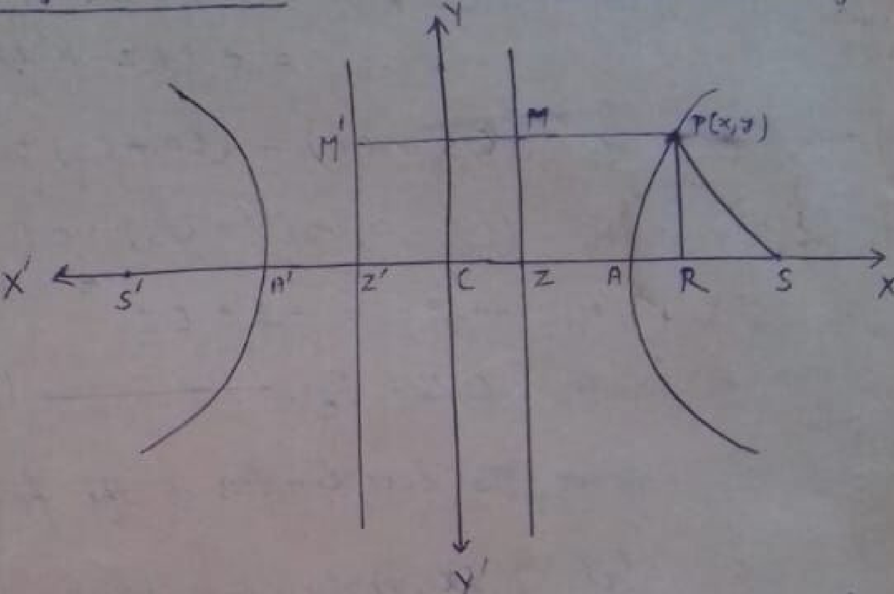
This is the standard form of the equation of ellipse.

Hyperbola

① Define a hyperbola and obtain its equation in standard form:

A hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point in the plane to its distance from a fixed straight line in the same plane is constant and greater than unity. The fixed point is called the focus, the fixed st. line is called the directrix & the constant ratio is called the eccentricity

Eqⁿ of the hyperbola :-



Let S be the focus, ZM the directrix and e(>1) the eccentricity of the hyperbola

Let SZ be perpendicular to the directrix

We divide SZ internally at A and externally at A' in the ratio of e : 1

$$\therefore \frac{AS}{AZ} = \frac{A'S}{A'Z} = \frac{e}{1} \quad \text{——— (1)}$$

By the definition of hyperbola, it is clear that the points A & A' are on the hyperbola.

Let AA' = 2a & C be the middle point of AA'.

$$\therefore CA = CA' = a \quad \text{——— (2)}$$

Let us take C as origin, CA as x-axis and CY, perpendicular to CA, as y-axis

From (1), $AS = e.AZ$ and $A'S = e.A'Z$

$\therefore AS + A'S = e(AZ + A'Z)$
$= e.AA'$
$= e.2a$

or, $(CS - CA) + (CA' + CS) = 2ae$

or, $(CS - a) + (a + CS) = 2ae$ [From (2)]

or, $2CS = 2ae$

or $CS = ae$ ——————— (3)

Also, $AS - A'S = e.AZ - e.A'Z$
$= e(AZ - A'Z)$

or, $(CS - CA) - (CA' + CS) = e[(CA - CZ) - (CA' + CZ)]$

or, $(\cancel{CS} - a) - (a + \cancel{CS}) = e(CA - CZ - CA' - CZ)$
$= e(\cancel{a} - 2CZ - \cancel{a})$

or, $-2a = -2e.CZ$

or, $CZ = \frac{a}{e}$ ——————— (4)

~~Hence~~ The coordinates of the focus S are $(ae, 0)$

Let $P(x, y)$ be any point on the hyperbola & PM be perpendicular on the directrix. We also draw a perpendicular PR on ZX $\therefore CR = x$ & $PR = y$

By definition of the hyperbola,

$\frac{PS}{PM} = e$ or, $PS = e.PM$

or, $PS^2 = e^2.PM^2 = e^2.RZ^2 = e^2(RC - CZ)^2$

or, $(x - ae)^2 + (y - 0)^2 = e^2(x - \frac{a}{e})^2$

or, $x^2 - 2aex + a^2e^2 + y^2 = e^2(x^2 - 2x\frac{a}{e} + \frac{a^2}{e^2})$

or, $x^2 - \cancel{2aex} + a^2e^2 + y^2 = e^2x^2 - \cancel{2aex} + a^2$

or, $x^2(1-e^2) + y^2 = a^2(1-e^2)$

or, $\frac{x^2(1-e^2)}{a^2(1-e^2)} + \frac{y^2}{a^2(1-e^2)} = 1$

or, $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ or, $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

or, $\frac{x^2}{a^2} - \frac{y}{b^2} = 1$, where $a^2(e^2-1) = b^2$

Proof :-

Let the general equation of second degree be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{——— (1)}$$

Let the axes of coordinates be rectangular.

We now turn the axes through a suitable angle θ so that the term in xy (or, the coefficient of xy) in the transformed eq^n of (1) vanishes.

Let the transformed equation of (1) becomes

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c = 0 \quad \text{——— (2)}$$

So that $ax^2 + 2hxy + by^2 = a'x'^2 + b'y'^2$ [The term xy vanishes in the transformed eq^n]

From the theory of invariants, we have

$$a + b = a' + b' \quad \text{——— (3)}$$

$$\& \quad ab - h^2 = a'b' \quad \text{——— (4)}$$

Now we discuss ~~three~~ two cases i.e. when $ab - h^2 = 0$ & when $ab - h^2 \neq 0$

Case I When $ab - h^2 = 0$

Then from (4),

$$a'b' = 0$$

$\therefore$ Either $a' = 0$ or $b' = 0$

Let $a' = 0$. Then the eq^n (2) becomes

$$b'y'^2 + 2g'x' + 2f'y' + c = 0$$

This represents a parabola whose axis is parallel to x-axis [or, since the highest degree terms form a perfect square] [[illegible]]

Case II When $ab - h^2 \neq 0$.

Then from (4),

$$a'b' \neq 0$$

$\therefore a' \neq 0$ and $b' \neq 0$

The eqn (2) can be written as

$$a'\left(x'^2 + \frac{2g'}{a'}x'\right) + b'\left(y'^2 + \frac{2f'}{b'}y'\right) + c = 0$$

or, $$a'\left(x' + \frac{g'}{a'}\right)^2 + b'\left(y' + \frac{f'}{b'}\right)^2 = \frac{g'^2}{a'} + \frac{f'^2}{b'} - c$$

or, $$\frac{\left(x' + \frac{g'}{a'}\right)^2}{b'} + \frac{\left(y' + \frac{f'}{b'}\right)^2}{a'} = \frac{1}{a'b'}\left(\frac{g'^2}{a'} + \frac{f'^2}{b'} - c\right),$$

which is of the form $\frac{\left(x' + \frac{g'}{a'}\right)^2}{B} + \frac{\left(y' + \frac{f'}{b'}\right)^2}{A} = 1$,

where B and A have the same or different signs according as b' & a' have the same or different signs.

Therefore it follows that (2) represents an ellipse with its centre at $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ provided a' & b' have the same sign i.e. $a'b' > 0$ i.e. $ab - h^2 > 0$ [from (4)]

Again (2) represents a hyperbola with its centre at $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ provided a' & b' have different signs i.e. $a'b' < 0$ i.e. $ab - h^2 < 0$ [from (4)]

Thus the equation (1) represents

a parabola if $ab - h^2 = 0$,

an ellipse if $ab - h^2 > 0$

& a hyperbola if $ab - h^2 < 0$.