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For T.D.C. Part III

Paper-6

Gr— [redacted] A

(1)

Automorphisms of a group

Definition. (**Madras 1983; Meerut 78; Kanpur 86; B.H.U. 87**)

An isomorphic mapping of a group G onto itself is called an automorphism of G.

Thus $f : G \xrightarrow[\text{one-one}]{\text{onto}} G$ is an automorphism of G if

$$f(ab) = f(a)\, f(b) \;\forall\; a, b \in G.$$

Solved Examples

Ex. 1. *Show that the mapping*

$$f : \mathbf{I} \to \mathbf{I} \text{ such that } f(x) = -x \;\forall\; x \in \mathbf{I}$$

is an automorphism of the additive group of integers **I**.

Solution. Obviously the mapping f is one-one onto.

Let x_1, x_2 be any two elements of **I**. Then

Theorem. *The set of all automorphisms of a group forms a group with respect to composite of functions as the composition.*

(Meerut 1989; Gujrat 71; Kanpur 86; Madurai 88; Raj. 77)

Proof. Let $A(G)$ be the collection of all automorphisms of a group G. Then $A(G)=\{f : f \text{ is an automorphism of } G\}$.

We shall prove that $A(G)$ is a group with respect to composite of functions as composition.

Closure property. Let $f, g \in A(G)$. Then f, g are one-one mappings of G onto itself. Therefore gf is also a one-one mapping of G onto itself. If a, b be any two elements of G, we have

$$(gf)(ab)=g[f(ab)]=g[f(a)f(b)]$$
$$=g[f(a)]\,g[f(b)]=[(gf)(a)]\,[(gf)(b)].$$

$\therefore$ gf is also an automorphism of G. Thus $A(G)$ is closed with respect to composite composition.

Associativity. We know that composite of arbitrary mappings is associative. Therefore composite of automorphisms is also associative.

Existence of Identity. The identity function i on G is also an automorphism of G. Obviously i is one-one onto and if $a, b \in G$, then $i(ab)=ab=i(a)\,i(b)$. Thus $i \in A(G)$ and if $f \in A(G)$, we have $if=f=fi$.

Existence of Inverse. Let $f \in A(G)$. Since f is a one-one mapping of G onto itself, therefore f^{-1} exists and is also a one-one mapping of G onto itself. We shall show that f^{-1} is also an automorphism of G. Let $a, b \in G$. Then there exist $a', b' \in G$ such that

$$f^{-1}(a)=a' \Leftrightarrow f(a')=a$$
$$f^{-1}(b)=b' \Leftrightarrow f(b')=b.$$

We have $f^{-1}(ab)=f^{-1}[f(a')f(b')]$

$$=f^{-1}[f(a'b')]=a'b'=f^{-1}(a)\,f^{-1}(b).$$

$\therefore$ f^{-1} is an automorphism of G and thus

$$f \in A(G) \Rightarrow f^{-1} \in A(G).$$

Therefore each element of $A(G)$ possesses inverse.

$f(x_1+x_2)=-(x_1+x_2)=(-x_1)+(-x_2)=f(x_1)+f(x_2)$.

Hence f is an automorphism of **I**.

Ex. 2. *Show that $a \to a^{-1}$ is an automorphism of a group G iff G is abelian.* **[Nagarjuna 1978; Madras 78; Meerut 82, 83, 84, 88]**

Solution. Let $f: G \to G$ be such that $f(x)=x^{-1}\ \forall\ x \in G$.

The function f is one-one because

$f(x)=f(y) \Rightarrow x^{-1}=y^{-1} \Rightarrow (x^{-1})^{-1}=(y^{-1})^{-1} \Rightarrow x=y$.

Also if $x \in G$, then $x^{-1} \in G$ and we have $f(x^{-1})=(x^{-1})^{-1}=x$.

$\therefore$ f is onto.

Now suppose G is abelian. Let a, b be any two elements of G. Then $f(ab)=(ab)^{-1}$ [by def. of f]

$=b^{-1}a^{-1}=a^{-1}b^{-1}$ [$\because$ G is abelian]

$=f(a)f(b)$ [by def. of f]

$\therefore$ f is an automorphism of G.

Conversely suppose that f is an automorphism of G. Let $a, b \in G$.

We have $f(ab)=(ab)^{-1}$ [by def. of f]

$=b^{-1}a^{-1}=f(b)f(a)$ [by def. of f]

$=f(ba)$. [$\because$ f is an automorphism]

Since f is one-one, therefore

$f(ab)=f(ba) \Rightarrow ab=ba \Rightarrow G$ is abelian.

Theorem 3. *The set $I(G)$ of all inner automorphisms of a group G is a normal subgroup of the group of its automorphisms isomorphic to the quotient group G/Z of G where Z is the centre of G.*

(I. A. S. 1970, 88; Delhi 70; Nagarjuna 78; Madurai 88; B.H.U. 88; Gujrat 71; Dibrugarh 78; Meerut 74, 78, 79; G.N.D.U. Amritsar 87)

Proof. Let $A(G)$ denote the group of all automorphisms of G. Then $I(G) \subseteq A(G)$.

Let $a, b \in G$. We shall first prove the following two results :

(i) $f_{a^{-1}} = f_a^{-1}$ i.e.. the inner automorphism $f_{a^{-1}}$ is the inverse function of the inner automorphism f_a.

(ii) $f_a f_b = f_{ba}$.

Proof of (i). If $x \in G$, then we have

$$(f_a f_{a^{-1}})(x) = f_a[f_{a^{-1}}(x)] = f_a[(a^{-1})^{-1} xa^{-1}] = f_a[axa^{-1}]$$
$$= a^{-1}(axa^{-1})a = x.$$

$\therefore$ $f_a f_{a^{-1}}$ is the identity function on G.

$$\therefore \quad f_{a^{-1}} = (f_a)^{-1}.$$

Proof of (ii). If $x \in G$, then we have

$$(f_a f_b)(x) = f_a[f_b(x)] = f_a(b^{-1}xb) = a^{-1}(b^{-1}xb)a = (a^{-1}b^{-1})x(ba)$$
$$= (ba)^{-1}x(ba) = f_{ba}(x).$$

$$\therefore \quad f_a f_b = f_{ba}.$$

Now we shall prove that $I(G)$ is a subgroup of $A(G)$. Let f_a, f_b be any two elements of $I(G)$. Then

$$f_a(f_b)^{-1} = f_a f_{b^{-1}} = f_{b^{-1}a} \in I(G) \text{ since } b^{-1}a \in G.$$

Thus $f_a, f_b \in I(G) \Rightarrow f_a(f_b)^{-1} \in I(G)$.

$\therefore$ $I(G)$ is a subgroup of $A(G)$.

Now we shall prove that $I(G)$ is a normal subgroup of $A(G)$.

Let $f \in A(G)$ and $f_a \in I(G)$. If $x \in G$, then we have

$$(f f_a f^{-1})(x) = (f f_a)[f^{-1}(x)] = f[f_a(f^{-1}(x))]$$
$= f[a^{-1} f^{-1}(x)a]$
$= f(a^{-1}) f[f^{-1}(x)] f(a)$ [$\because$ f is composition preserving]
[$\because$ $f[f^{-1}(x)] = x$]
$= f(a^{-1}) xf(a)$
$= [f(a)]^{-1} xf(a)$
$= c^{-1}xc$ where $f(a) = c \in G$
$= f_c(x)$.

$\therefore$ $f f_a f^{-1} = f_c \in I(G)$ since $c \in G$.

$\therefore$ $I(G)$ is a normal subgroup of $A(G)$.

Now we shall show that $I(G)$ is isomorphic to G/Z. For this we shall show that $I(G)$ is a homomorphic image of G and Z is

Therefore $A(G)$ is a group with respect to composite composition.

§ 8. **Inner Automorphisms**. We shall now study a special type of automorphisms known as **inner automorphisms**. First we shall prove a preliminary theorem.

Theorem 1. *Let a be a fixed element of a group G. Then the mapping $f_a : G \to G$ defined by $f_a(x) = a^{-1}xa \ \forall \ x \in G$ is an automorphism of G.* **(Gujrat 1971)**

Proof. The mapping f_a is one-one. Let x, y be any two elements of G. Then

$f_a(x) = f_a(y) \Rightarrow a^{-1}xa = a^{-1}ya \Rightarrow x = y$, by cancellation laws in G. Therefore the mapping f_a is one-one.

The mapping f_a is also onto G. If y is any element of G, then $aya^{-1} \in G$ and we have $f_a(aya^{-1}) = a^{-1}(aya^{-1})a = y$.

$\therefore$ f_a is onto G.

Finally if $x, y \in G$ then $f_a(xy) = a^{-1}(xy)a = (a^{-1}xa)(a^{-1}ya)$ $= f_a(x) f_a(y)$. Hence f_a is an automorphism of G.

Inner Automorphism. Definition.

If G is a group, the mapping

$$f_a : G \to G \text{ defined by } f_a(x) = a^{-1}xa \ \forall \ x \in G$$

is an automorphism of G known as inner automorphism.

(Delhi 1988; Nagarjuna 78; B.H.U. 87, 88)

Also an automorphism which is not inner is called **an outer automorphism.**

Theorem 2. *For an abelian group the only inner automorphism is the identity mapping whereas for non-abelian groups there exist non-trivial automorphisms.* **(Raj. M. Sc. 1966)**

Proof. Suppose G is an abelian group and f_a is an inner automorphism of G. If $x \in G$, we have

$$f_a(x) = a^{-1}xa = a^{-1}ax \qquad [\because G \text{ is abelian}]$$
$$= ex = x.$$

Thus $\qquad f_a(x) = x \ \forall \ x \in G.$

$\therefore$ f_a is the identity mapping of G.

Let now G be non-abelian. Then there exist at least two elements say $a, b \in G$ such that

$$ba \neq ab \Rightarrow a^{-1}ba \neq b \Rightarrow f_a(b) \neq b.$$

Hence f_a is not the identity mapping of G. Thus for non-abelian groups there always exist non-trivial inner automorphisms.

the kernel of the corresponding homomorphism.

Then by the fundamental theorem on homomorphism of groups we shall have $G/Z \cong I(G)$.

Consider the mapping $\phi : G \to I(G)$ defined by

$$\phi(a) = f_{a^{-1}} \;\forall\; a \in G.$$

Obviously ϕ is onto $I(G)$ because $f_a \in I(G) \Rightarrow a \in G$ and this implies $a^{-1} \in G$.

Now $$\phi(a^{-1}) = f_{(a^{-1})^{-1}} = f_a.$$

$\therefore$ ϕ is onto $I(G)$.

Now to prove that $\phi(ab) = \phi(a)\phi(b) \;\forall\; a, b \in G$.

We have $\phi(ab) = f_{(ab)^{-1}} = f_{b^{-1}a^{-1}} = f_{a^{-1}} f_{b^{-1}} = \phi(a)\phi(b)$.

Now to show that Z is the kernel of ϕ.

The identity function i on G is the identity of the group $I(G)$.

Let K be the kernel of ϕ.

Then we have $z \in K \Leftrightarrow \phi(z) = i \Leftrightarrow f_{z^{-1}} = i \Leftrightarrow f_{z^{-1}}(x) = i(x)$

$\forall\; x \in G \Leftrightarrow (z^{-1})^{-1} x z^{-1} = x \;\forall\; x \in G \Leftrightarrow z x z^{-1} = x \;\forall\; x \in G$

$\Leftrightarrow zx = xz \;\forall\; x \in G \Leftrightarrow z \in G$.

$\therefore$ $K = Z$.

Hence the theorem.