## Maxwell Boltzmann distribution

Let total no. of particles be N and all these particles are divided into groups containing $n_1, n_2, n_3 \dots n_i$ no. of particles. The no. of ways in which it can be done will be:

→ Probability of macrostates = ${}^{N}C_{n_1} \; {}^{N-n_1}C_{n_2} \; {}^{N-n_1-n_2}C_{n_3} \dots {}^{N-n_1-\dots n_{i-1}}C_{n_i}$

$$= \frac{N!}{n_1!(N-n_1)!} \frac{(N-n_1)!}{n_2!(N-n_1-n_2)!} \dots$$

$$= \frac{N!}{n_1! \, n_2! \, n_3! \dots}$$

If $g_1, g_2, g_3 \cdots g_i$ be the no. of cells that are available to ~~those~~ the particles $n_1, n_2, n_3 - n_i$, ~~[illegible]~~ and if there is no restriction on no. of ways the particles can be arranged in the ~~$g_i$~~ $g_i$ cells, then we have Probability of microstates ~~[illegible]~~ to this macrostate will be:

$$\text{No. of way} = (g_1)^{n_1} (g_2)^{n_2} \cdots (g_i)^{n_i}$$

$\therefore$ Total no. of ways will be

$$\Omega = N! \prod_i \frac{(g_i)^{n_i}}{n_i!} \longrightarrow \text{Total no. of ways.}$$

We know that the equilibrium condition is given as:

$$d(\ln \Omega) = 0.$$

$\therefore$ we can be

$$\ln \Omega = \ln N! + \sum_i n_i \ln g_i - \sum_i \ln n_i!$$

$$\ln n_i! = n_i \ln n_i - n_i$$

$$\therefore \ln \Omega = N \ln N - N + \sum_i [n_i \ln g_i - n_i \ln n_i + n_i]$$

Taking differential:

$$\Rightarrow d \ln \Omega = \sum_i [\ln g_i dn_i - \ln n_i dn_i - dn_i + dn_i]$$

$$\Rightarrow d \ln \Omega = \sum_i [\ln g_i dn_i - \ln n_i dn_i]$$

$$\Rightarrow d \ln \Omega = \sum_i \left[\ln\left(\frac{g_i}{n_i}\right) dn_i\right]$$

We know the ~~rest~~ assumptions are:

$$\sum dn_i = 0, \quad \sum \epsilon_i dn_i = 0, \quad \sum d \ln \Omega = 0.$$

$\therefore$ Using Langrange's method of undetermined multiplier, we get

$$d \ln \Omega + \alpha \sum dn_i - \beta \sum \epsilon_i dn_i = 0$$

①+②-2:

$$\Rightarrow \sum_i \left[\ln\left(\frac{g_i}{n_i}\right) + \alpha - \beta \epsilon_i\right] dn_i = 0$$

It holds for all $i$,

$$\therefore \ln\left(\frac{g_i}{n_i}\right) + \alpha - \beta \epsilon_i = 0$$

$$\Rightarrow \boxed{\frac{n_i}{g_i} = e^{\alpha - \beta \epsilon_i}}$$

$$\Rightarrow n_i = g_i e^{\alpha - \beta \epsilon_i} = g_i A e^{-\beta \epsilon_i}$$

Total no of particles $= \sum n_i = N$.

$$\Rightarrow \sum g_i A e^{-\beta \epsilon_i} = N$$

$$\Rightarrow A = \frac{N}{\sum g_i e^{-\beta \epsilon_i}}$$

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$\therefore\ n_i = \frac{N}{Z} g_i e^{-\beta \epsilon_i}$ where $Z = \sum g_i e^{-\beta \epsilon_i}$ = partition function

The probability of occupancy of energy level $\epsilon_i$ will be.

$$P(\epsilon_i) = \frac{n_i}{N} = \frac{g_i e^{-\beta \epsilon_i}}{\sum g_i e^{-\beta \epsilon_i}}$$

$\beta$:

$$S = k_B \ln \Omega$$

$$dS = k_B\, d(\ln \Omega)$$

$$T\,dS = k_B T\, d(\ln \Omega).$$

$$\Rightarrow dE = \sum \epsilon_i dn_i = KT\, d(\ln \Omega)$$

$$\text{or } \sum \epsilon_i dn_i = KT \sum (+\alpha - \beta \epsilon_i) dn_i$$

$$= -\alpha KT \cancel{\sum dn_i}^{\,0} + KT\beta \sum \epsilon_i dn_i$$

$$\Rightarrow \beta = 1/KT.$$

$\therefore$ for MB:

$$\boxed{\frac{n_i}{g_i} = \frac{N g_i e^{-\epsilon_i/KT}}{\sum g_i e^{-\epsilon_i/KT}}}$$

$$P(\epsilon_i) = \frac{g_i e^{-\epsilon_i/KT}}{\sum g_i e^{-\epsilon_i/KT}}.$$

Z: partition fn:

$$\langle x \rangle = \frac{\sum x P(x)}{\sum P(x)} = \frac{\int_{-\infty}^{\infty} x P(x) dx}{\int_{-\infty}^{\infty} P(x) dx}$$

If not given, take $g_i = 1$.

Q. 100 particles; 3 groups; $g = 1$.

| | | |
|---|---|---|
| $\epsilon_2 = 2KT$ | $n_2$ | $g_2 = 1$ |
| $\epsilon_1 = KT$ | $n_1$ | $g_1 = 1$ |
| $0$ | $n_0$ | $g_0 = 1$ |

1 cell per energy level.

$$n_0 = Ae^{-\beta \epsilon_i} = A$$

$$n_1 = Ae^{-\beta \epsilon_1} = Ae^{-1}$$

$$n_2 = Ae^{-2}$$

$$N = A[1 + e^{-1} + e^{-2}] = 100$$

$$\Rightarrow A = \frac{100}{1 + e^{-1} + e^{-2}} = 72.$$

$\therefore n_0 = 72,\ n_1 = 26,\ n_2 = 2$

$\Rightarrow P(\epsilon_0) = 0.72,\ P(\epsilon_1) = 0.26,\ P(\epsilon_2) = 0.02$

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Total energy

$$E = \sum \epsilon_i N_i = \sum 0 \times 72 + KT \times 2 G + 2KT \times 2$$
$$= 30KT$$

$$\langle E \rangle = \frac{\sum \epsilon_i P(\epsilon_i)}{\sum P(\epsilon_i)} = \frac{E}{N} = 0.3\,KT$$

## MB distribution

$$f(\epsilon_i) = \frac{n_i}{g_i} = Ae^{-\beta \epsilon_i}$$

No. of particles within energy level b/w $\epsilon_i$ and $\epsilon_i + d\epsilon_i$ will be:

$$n(\epsilon)d\epsilon$$

$\therefore$ we have,

$$n(\epsilon)d\epsilon = Ae^{-\epsilon/KT} g(\epsilon)d\epsilon \rightarrow$$ no. of particles with energy b/w $\epsilon$ and $\epsilon + d\epsilon$ at eqm temp.

We recall that $g(\epsilon)d\epsilon$ = density of states.

Total no. of particles: N

$$\Rightarrow N = \int n(\epsilon)d\epsilon$$
$$= A\int e^{-\epsilon/KT} \cdot \frac{2\pi V}{h^3}(2m)^{3/2} \epsilon^{1/2} d\epsilon$$
$$= B\int e^{-\epsilon/KT} \epsilon^{1/2} d\epsilon$$

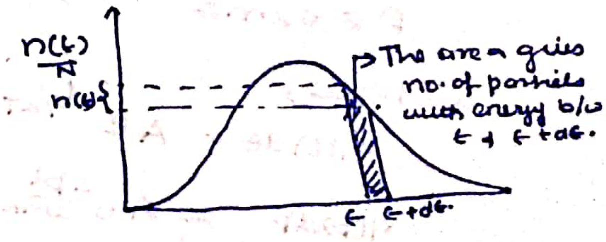
put $\epsilon/KT = x$

$$\Rightarrow N = B(KT)^{3/2}\int e^{-x} x^{1/2} dx$$
$$= B(KT)^{3/2}\,\Gamma(3/2).$$

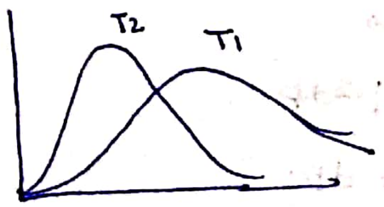
$$\therefore B = \frac{2N}{(KT)^{3/2}\sqrt{\pi}}$$

$$\text{or } B = \frac{2\pi N}{(\pi KT)^{3/2}}.$$

$$\therefore \boxed{n(\epsilon)\,d\epsilon = \frac{2\pi N}{(\pi KT)^{3/2}} \sqrt{\epsilon}\, e^{-\epsilon/KT} d\epsilon.}$$ M-B distribution formula.




$T_1 > T_2$:



Most probable energy

$$\frac{dn}{d\epsilon} = 0$$

$$\frac{d}{d\epsilon}\left[\epsilon^{1/2} e^{-\epsilon/kT}\right] = 0$$

$$\left[\frac{1}{2\sqrt{\epsilon}} - \frac{\epsilon^{1/2}}{kT}\right] = 0 \qquad \epsilon^{3/2+1/2}$$

$$\text{or} \cdot \boxed{\frac{kT}{2} = \epsilon}$$

Mean energy:

$$\langle \epsilon \rangle = \frac{\int_0^\infty \epsilon\, n(\epsilon) d\epsilon}{\int_0^\infty n(\epsilon) d\epsilon}$$

$$= \frac{\int \epsilon^{3/2} e^{-\epsilon/kT} d\epsilon}{\int \epsilon^{1/2} e^{-\epsilon/kT} d\epsilon}$$

$$= \frac{(kT)^{5/2}}{(kT)^{3/2}} \frac{\int x^{3/2} e^{-x} dx}{\int e^{1/2} e^{-x} dx}$$

$$= kT \frac{\Gamma(5/2)}{\Gamma(3/2)}$$

$$= \frac{3}{2} kT.$$

$$\therefore \boxed{\langle \epsilon \rangle = \frac{3}{2} kT}$$

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = \underbrace{\frac{\langle p_x^2 \rangle}{2m} + \frac{\langle p_y^2 \rangle}{2m} + \frac{\langle p_z^2 \rangle}{2m}}_{\text{Each degree of freedom contributes } \frac{1}{2}kT}$$

Momentum distribution:

$$p^2 = 2m\epsilon.$$

$$p = \sqrt{2m\epsilon}$$

we have,

$$n(\epsilon) d\epsilon = A e^{-\epsilon/kT} g(\epsilon) d\epsilon$$

$$n(p) dp = A e^{-\frac{p^2}{2mkT}} \cdot \frac{4\pi V}{h^3} p^2 dp.$$

$$\text{or } n(p)\, dp = B p^2 e^{-p^2/2mkT} dp.$$

$$N = B \int p^2 e^{-p^2/2mkT} dp.$$

$$= \frac{4\pi N}{(2\pi mkT)^{3/2}}$$

∴ we get,

$$\boxed{n(p)dp = \frac{4\pi N}{(2\pi mkT)^{3/2}} p^2 e^{-p^2/2mkT} dp}$$

Velocity distribution,

$p = \cancel{\quad} mv$

$$n(v)dv = \frac{4\pi N}{(2\pi mkT)^{3/2}} \frac{1}{4} \cancel{\quad} m^2 v^2 e^{-\frac{mv^2}{2kT}} dv$$

$$\text{or } n(v)dv = \frac{4\pi N m^3}{(2\pi mkT)^{3/2}} v^2 e^{-\frac{mv^2}{2kT}} dv.$$

$$= 4\pi N \left(\frac{m}{2\pi kT}\right) v^2 e^{-mv^2/2kT} dv$$

We can also start with

$$n(v)dv = A e^{-\frac{mv^2}{2kT}} \frac{4\pi V}{h^3} v^2 dv.$$

& we get the same result.

We have,

$$n(v)dv = 4\pi N \left(\frac{m}{2\pi kT}\right) v^2 e^{-mv^2/2kT} dv.$$

We can calculate:

→ Most probable speed : $\frac{dn(v)}{dv} = 0 \equiv v_p = \sqrt{\frac{2kT}{m}}$

→ RMS speed $= \sqrt{\frac{\int v^2 n(v) dv}{\int n(v) dv}} \Rightarrow v_{rm} = \sqrt{\frac{3kT}{m}}$

→ Mean speed $= \frac{\int v\, n(v) dv}{\int n(v) dv} \Rightarrow \bar{v} = \sqrt{\frac{8kT}{\pi m}}.$

Probability distribution:

$$P(v)dv = \frac{n(v)}{N} dv$$

$$= P(v_x) P(v_y) P(v_z)\, dv_x\, dv_y\, dv_z$$

$$= P(v_x) P(v_y) P(v_z)\, 4\pi v^2 dv.$$

$$= P(v_x) P(v_y) P(v_z)\, e^{-\frac{m}{2kT}(v_x^2+v_y^2+v_z^2)} dv$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{m}{2kT}}$$

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Fermi Dirac Statistics : Fermions are governed by Pauli's exclusion principle. Therefore, no two particles can occupy the same sublevel, & so no. of levels (cells) must be greater than no. of fermions (i.e. $g_i > n_i$).

No. of ways in which $n_i$ fermions can occupy $g_i$ sublevels is equal to no. of ways in which $n_i$ things can be taken at a time from $g_i$ different things. ~~occup~~

∴ we get,

$$g_iC_{n_i} = \frac{g_i!}{n_i!\,(g_i-n_i)!}$$

[Note that 2 spin states are actually 2 different cells ⇒ $g$ (case) is multiplied by 2]

Therefore, for other sublevels and considering them all, we get,

$$\Omega_{FD} = P_{FD} = \prod_i \frac{g_i!}{n_i!\,(g_i-n_i)!} \quad \text{—①}$$

For equilibrium at any temp T, we must have

$$dS = 0$$

We know that,

$$S = k_B \ln\Omega$$

$$dS = 0 \Rightarrow k_B\, d(\ln\Omega) = 0$$

Taking log of eqn ①:

$$\ln\Omega_{FD} = \sum_i [\ln g_i! - \ln n_i! - \ln(g_i-n_i)!]$$

$$= \sum_i [g_i\ln g_i - g_i - n_i\ln n_i + n_i - (g_i-n_i)\ln(g_i-n_i) + g_i - n_i]$$

$$d(\ln\Omega_{FD}) = \sum_i [-\cancel{dn_i} - \ln n_i\, dn_i + \cancel{dn_i} + \cancel{dn_i} + \ln(g_i-n_i)\,dn_i - \cancel{dn_i}]$$

$$d(\ln\Omega_{FD}) = \sum \ln\left(\frac{g_i-n_i}{n_i}\right) dn_i$$

We also know that, $\sum dn_i = 0$ & $\sum \epsilon_i dn_i = 0$

Doing ① + ③ − ②: Lagrange's method of undetermined multipliers;

$$\sum \delta n_i \left[\ln\left(\frac{g_i-n_i}{n_i}\right) + \alpha - \beta\epsilon_i\right] = 0$$

It holds good for all 'i', we get,

$$\ln\left(\frac{g_i-n_i}{n_i}\right) = -(\alpha - \beta\epsilon_i)$$

$$\frac{g_i}{n_i} - 1 = e^{-(\alpha-\beta\epsilon_i)}$$

$$\frac{g_i}{n_i} = 1 + e^{-(\alpha-\beta\epsilon_i)}$$

$$\Rightarrow n_i/g_i = \frac{1}{e^{-(\alpha-\beta\epsilon_i)} + 1}.$$

$$f(\epsilon_i) = \frac{n_i}{g_i} = \frac{1}{1 + e^{-(\alpha - \beta\epsilon_i)}}$$

We have, $S = k \ln \Omega$

$\Rightarrow T\,dS = k_B T \ln \Omega$

$$= k_B T \sum -(\alpha - \beta\epsilon_i)\, dn_i$$

or $\sum \epsilon_i\, dn_i = k_B T\, \beta \sum \epsilon_i\, dn_i$

$\Rightarrow \boxed{\beta = \frac{1}{k_B T}}$

Putting $\epsilon_F / kT = \alpha_0$ :

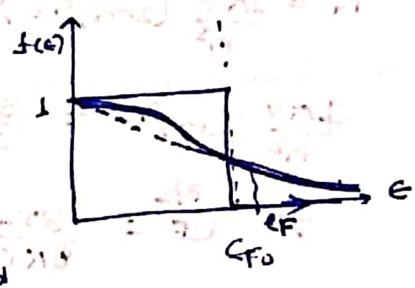
$$\Rightarrow f(\epsilon_i) = \frac{n_i}{g_i} = \frac{1}{1 + e^{\frac{(\epsilon_i - \epsilon_F)}{kT}}}$$

∴ General formula of distribution:

$$f(\epsilon) = \frac{n(\epsilon)}{g(\epsilon)} = \frac{1}{1 + e^{(\epsilon - \epsilon_F)/kT}}$$

↓

less than 1.




At $T = 0K$,

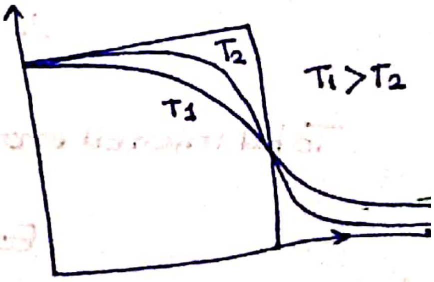
1) $\epsilon > \epsilon_F$, $f(\epsilon) = 0 \rightarrow$ no state is occupied

2) $\epsilon < \epsilon_F$, $f(\epsilon) = 1 \rightarrow$ all states equally occupied

→ complete definition of $\epsilon_F$:

① characteristics of fermions at temp T.

② 0K, energy below which all states are fully occupied & above which no state is occupied.




→ Note that:

$f = \frac{1}{2}$ @ $\epsilon = \epsilon_F$ and not $\epsilon_{F_0}$. $\epsilon_F$ lies close to but not at $\epsilon_{F_0}$.

<u>complete degenerate $T = 0K$.</u>

The occupation no. for sth energy level is given by in a FD system,

$$\bar{n}_{FD} = \frac{1}{e^{(\epsilon_s - \mu)/k_B T} + 1}. \quad \text{where } \mu \equiv \text{chemical potential}$$

For dealing with macroscopic system, we replace $\bar{n}_{FD}$ with

$$f_{FD} = \frac{1}{e^{\frac{(\epsilon - \mu)}{k_B T}} + 1}$$

To find ~~Fermi level~~ ($\mu = \epsilon_F / k_B T$):

We know that

$$N = \int_0^\infty \iint f_{FD}(\epsilon)\, \underbrace{\frac{d^3r\, d^3p}{h^3}}_{\text{DOS in interval } \epsilon \text{ and } \epsilon+d\epsilon}$$

$$N = \frac{2\pi V}{h^3}(2m)^{3/2} \int_0^\infty \bar{n}_{FD}(\epsilon)\, \epsilon^{1/2}\, d\epsilon .$$

We know that,

$$f(\epsilon) = 1 \qquad \epsilon < \epsilon_F \qquad \epsilon_F = \mu\,(T=0)$$

$$= 0 \qquad \epsilon > \epsilon_F \qquad \hookrightarrow \text{Fermi energy}.$$

$$\Rightarrow \quad N = \frac{2\pi V}{h^3}(2m)^{3/2}\int_0^{\epsilon_F} \epsilon^{1/2} d\epsilon$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2}\, \frac{\epsilon_F^{3/2}}{3/2}$$

$\because$ • Spin $s = 1/2$, degeneracy $= 2s+1 = 2$.

$$\therefore \boxed{N = 2\times\frac{2\pi V}{h^3}(2m)^{3/2} \times \frac{\epsilon_F^{3/2}}{3/2}} \quad \text{— (1)}$$

$$\frac{4\pi V}{h^3}(2m)^{3/2} = \frac{3}{2}\frac{N}{\epsilon_F^{3/2}}$$

$$\Rightarrow \epsilon_F^{3/2} = \frac{3n h^3}{8\pi (2m)^{3/2}}.$$

$$\epsilon_{F_0} = \frac{1}{2m}\left(\frac{3nh^3}{8\pi}\right)^{2/3}.$$

$$= \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}.$$

$$\therefore \boxed{\epsilon_0 = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}} \rightarrow \text{Remember.}$$

Total internal energy of completely degenerate FD gas:

↳ factor of 2 due to multiplicity factor (=1)

$$E_0 = \frac{4\pi V}{h^3}(2m)^{3/2}\int_0 \epsilon^{3/2} d\epsilon$$

$$= \frac{4\pi V}{h^3}(2m)^{3/2}\,\frac{2}{5}\,\epsilon_{F_0}^{5/2} \quad \text{— (2)} \quad = \boxed{\frac{3}{5} N\epsilon_F}$$

Mean energy per fermion:

$$\bar{\epsilon} = \frac{E_0}{N} = \frac{3}{5}\epsilon_F .$$

$$\Rightarrow \boxed{\bar{\epsilon} = \frac{3}{5}\epsilon_F} \rightarrow *$$

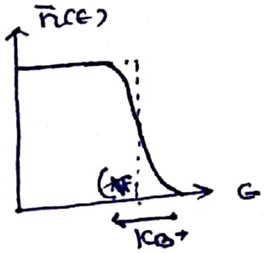
• we know that $C_V = \left(\frac{\partial E}{\partial T}\right)_V \Rightarrow$ specific heat of fermions drop to zero at absolute zero temperature i.e $C_V = 0$.

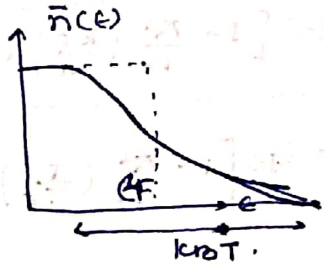
• Pressure;

$$p = \frac{2E}{3V} = \frac{2}{3} \times \frac{4\pi}{h^3} (2m)^{3/2} \times \frac{2}{5} \epsilon_{f0}^{5/2}.$$

$$\boxed{p = \frac{2}{5}\left(\frac{N}{V}\right)\epsilon_F}$$ — This pressure is very large and is compensated by the coulomb attraction of electrons by ions.

## Strongly degenerate FD systems $(T \ll T_F)$.




FD distribution system:

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\epsilon_F)}+1}.$$

Total no. of particles at $T > 0K$, we have,

$$N = \frac{4\pi V}{h^3}(2m)^{3/2}\int_0^\infty \frac{\epsilon^{1/2}\, d\epsilon}{1+e^{\frac{(\epsilon-\epsilon_F)}{k_BT}}}$$

$\epsilon/k_BT = x$

$\Rightarrow d\epsilon = (k_BT)\, dx$

$\epsilon_F/k_BT = x_0$

$$\therefore N = \frac{4\pi V}{h^3}(2m)^{3/2}\int_0^\infty \frac{(k_BT)^{3/2} x^{1/2}\, dx}{1+e^{x-x_0}}$$

$$= \frac{4\pi V}{h^3}(2m)^{3/2}(k_BT)^{3/2}\int_0^\infty \frac{x^{1/2}\, dx}{1+e^{(x-x_0)}}$$

Sommerfeld's Lemma:

$$\int_0^\infty f(x-x_0)\, x^s\, dx = \frac{x_0^{s+1}}{s+1}\left[1 + \frac{\pi^2}{6}\frac{(s(s+1))}{x_0^2} + \cdots\right]$$

Taking $f(x-x_0) = \frac{1}{e^{(x-x_0)}+1}$ and $s = 1/2$, we get,

$$\int_0^\infty \frac{x^{1/2}\, dx}{e^{x-x_0}+1} = \frac{2}{3}x_0^{3/2}\left[1 + \frac{\pi^2}{8x_0^2} + \cdots\right]$$

$$= \frac{2}{3}\left(\frac{\epsilon_F}{k_BT}\right)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{k_BT}{\epsilon_F}\right)^2 + \cdots\right]$$

$\therefore$ we can write:

$$N = \frac{2}{3}\frac{4\pi V}{h^3}(2m)^{3/2}(k_BT)^{3/2}\left(\frac{\epsilon_F}{k_BT}\right)^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{k_BT}{\epsilon_F}\right)^2+\cdots\right]$$

$$\text{or} \quad N = \frac{N}{\epsilon_{F0}^{3/2}}\,\epsilon_F^{3/2}\left(1+\frac{\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2\cdots\right)$$

$$\text{or} \quad \epsilon_F = \epsilon_{F0}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]^{-2/3}$$

$$\text{or} \quad \boxed{\epsilon_F = \epsilon_{F0}\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]}$$

$$\text{or} \quad \epsilon_F = \epsilon_{F0}\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\cdots\right]$$

Total energy

$$E = \int \epsilon\, dn_\epsilon$$

$$= \frac{4\pi V}{h^3}(2m)^{3/2}\int_0^\infty \frac{\epsilon^{3/2}\,d\epsilon}{e^{\frac{(\epsilon-\epsilon_F)}{k_BT}}+1}$$

$\epsilon/k_BT = x$, $\epsilon_F/k_BT = x_0$

$$E = \frac{4\pi V}{h^3}(2m)^{3/2}(k_BT)^{5/2}\int_0^\infty \frac{x^{3/2}\,dx}{e^{x-x_0}+1}$$

$$= \frac{3}{2}\frac{N}{(\epsilon_{F0})^{3/2}}(k_BT)^{5/2}\left(\frac{\epsilon_F}{k_BT}\right)^{5/2}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]$$

$$= \frac{3}{2}\frac{N}{(\epsilon_{F0})^{3/2}}\,\epsilon_F^{5/2}\times\frac{2}{5}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2\cdots\right]$$

$$= \frac{3}{5}N\frac{\epsilon_F^{5/2}}{\epsilon_{F0}^{3/2}}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]$$

$$\cancel{E = \frac{3}{5}N\epsilon_{F0}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]}$$

$$E = \frac{3}{5}N\epsilon_{F0}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2+\cdots\right]\left[1-\frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2\cdots\right]$$

$$\text{or} \quad \boxed{E = \frac{3}{5}N\epsilon_{F0}\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\cdots\right]}$$

Specific heat capacity

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{\pi^2}{2} N \epsilon_0 \left(\frac{T}{T_F^2}\right)$$

or $$\boxed{C_V = \frac{\pi^2}{2}\left(\frac{NK_B}{T_F}\right)T.}$$

$$\Rightarrow C_V = aT.$$

Entropy:
$$S = \int_0^T \frac{C_V dT}{T} = \frac{\pi^2}{2}\left(\frac{Nk}{T_F}\right)T.$$

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We have,
$$dn_E = \frac{4\pi V}{h^3}(2m)^{3/2} \frac{E^{1/2}\, dE}{e^{\frac{E-E_F}{kT}}+1}.$$

$$\frac{1}{2}mv^2 = E \Rightarrow mv\,dv = dE$$

$$\therefore dn_v = \frac{4\pi V}{h^3}(2m)^{3/2}\, mv \frac{\sqrt{m/2}\, v\, dv}{e^{\frac{m}{2kT}(v^2-v_F^2)}+1}$$

$$\text{or } dn_v = \frac{8\pi m^3 V}{h^3} v^2 \frac{dv}{e^{\frac{m}{2kT}(v^2-v_F^2)}+1}$$

$$\therefore \frac{dn_v}{dv} = \frac{8\pi m^3 V}{h^3} \frac{v^2}{e^{\frac{m}{2kT}(v^2-v_F^2)}+1}$$

For complete degenerate fermion, @ $T=0$

$$\Rightarrow \frac{dn_v}{dv} = \begin{cases} \frac{8\pi m^3 V}{h^3} v^2 & \text{or } V < V_F \\ 0 & V > V_F. \end{cases}$$

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Helmholtz function

$$F = E - TS$$
$$= \frac{3NE_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right] - \frac{\pi^2}{2}\left(\frac{NK}{T_F}\right)T^2$$
$$= \frac{3NE_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] - \frac{\pi^2}{2} N E_F \frac{T^2}{T_F^2}$$

or $$\boxed{F = \frac{3NE_F}{5}\left[1 - \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right]}$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{2}{3}\frac{E}{V} = \frac{2}{5}\frac{NE_F}{V}\left(1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right)$$