Vector Calculus

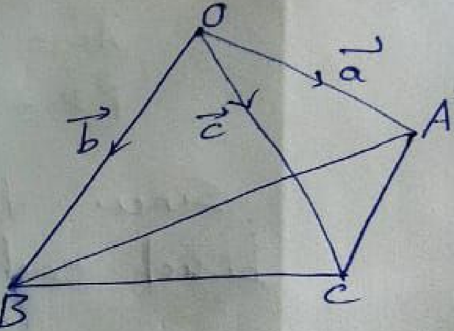
Area of a triangle whose vertices are known

Theorem Show that the vector area of a triangle whose vertices are the points $\vec{a}, \vec{b}, \vec{c}$ is given by

$$\boxed{\Delta = \frac{1}{2}\left(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}\right)}$$

Also deduce the condition of collinearity of three vectors $\vec{a}, \vec{b}, \vec{c}$.

Proof:- Consider a ΔABC where $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$; O being the origin, then



$$AB = \vec{b} - \vec{a}, \quad AC = \vec{c} - \vec{a}$$

$\therefore$ Vector area ΔABC

$$= \frac{1}{2}\left(\vec{AB} \times \vec{AC}\right)$$

$$= \frac{1}{2}(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$= \frac{1}{2}\left[\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} + \vec{a}\times\vec{a}\right]$$

(by distributive la

$$= \frac{1}{2}\left(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}\right) \quad \text{——} (1)$$

Proved

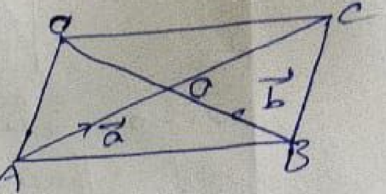
Next, if the points $A(\vec{a}), B(\vec{b}), C(\vec{c}$ are collinear, then

$$\Delta ABC = 0$$

$\therefore \boxed{\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b} = 0}$ using ① —— ②

This is the required condition of collinearity of $\vec{a}, \vec{b}, \vec{c}$.

Q.) Find the area of the parallelogram whose diagonals are $3\vec{i}+\vec{j}-2\vec{k}$, $\vec{i}-3\vec{j}+4\vec{k}$



We consider a parallelogram ABCD, whose diagonals are

$$\left.\begin{aligned}\vec{AC} &= \vec{a} = 3\vec{i}+\vec{j}-2\vec{k} \\ \vec{BD} &= \vec{b} = \vec{i}-3\vec{j}+4\vec{k}\end{aligned}\right\} \text{— ①}$$

Since the diagonals of a parallelogram bisect each other, we have

$$\left.\begin{aligned}\vec{OC} &= \frac{1}{2}\vec{AC} = \frac{1}{2}(3\vec{i}+\vec{j}-2\vec{k}) \\ \vec{OD} &= \frac{1}{2}\vec{BD} = \frac{1}{2}(\vec{i}-3\vec{j}+4\vec{k})\end{aligned}\right\} \text{— ②}$$

using ①

$$\therefore \vec{OC}\times\vec{OD} = \frac{1}{4}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \frac{1}{4}(-2\vec{i}-14\vec{j}-10\vec{k})$$

$$= \frac{1}{2}(-\vec{i}-7\vec{j}-5\vec{k}) \text{ —— ③}$$

Also $$|\vec{OC}\times\vec{OD}| = \frac{1}{2}\sqrt{(-1)^2+(-7)^2+(-5)^2}$$

$$= \frac{1}{2}\sqrt{1+49+25}$$

$$= \frac{5}{2}\sqrt{3} \text{ —— ④}$$

$\therefore$ Area of the parallelogram ABCD

$= 4(\Delta COD) = 4\left(\frac{1}{2}|\vec{OC} \times \vec{OD}|\right)$

$= 2\left(\frac{5}{2}\sqrt{3}\right) = 5\sqrt{3}$ sq units using 4