Uniform Continuity :-

We suppose f is continuous for every value of x in $[a, b]$. It means that if $x_0 \in [a, b]$ then given $\epsilon > 0$, there exists a positive number δ such that $|f(x) - f(x_0)| < \epsilon$ for every point x in the interval $]x_0 - \delta, x_0 + \delta[$. The number δ will depend upon x_0 as well as ϵ and so we may write it symbolically as $\delta(\epsilon, x_0)$. Now suppose that we keep ϵ fixed and vary x_0. Then for a given x_0 there will correspond a value δ. The set of values of δ corresponding to values of x_0 in $[a, b]$, may or may not have a non-zero lower bound. If this set of values of δ has a non-zero lower bound, say δ_0, then for every x_0 in $[a, b]$, we have $|f(x) - f(x_0)| < \epsilon$ for all x such that $|x - x_0| < \delta_0$. In such a case, we say that the function f is uniformly continuous in $[a, b]$.

Definition :- A function f is said to be uniformly continuous in $[a, b]$ iff for a given arbitrary small positive number ϵ, there can be found a number δ, depending only on ϵ, such that

$$x_1, x_2 \in [a, b] \text{ and } |x_1 - x_2| < \delta$$

$$\Rightarrow |f(x_1) - f(x_2)| < \epsilon$$

we have defined uniform continuity with reference to a finite closed interval $[a, b]$. But no such restriction is necessary for uniform continuity. The definition is essentially the same for intervals such as

$$]a, b[\;,\;]a, \infty[$$

It should, however, be noted carefully that uniform continuity is a property associated with an interval and not with a single point.

It follows from the definition of uniform continuity that for a given ϵ, the number δ should be such that the condition

$$|f(x_1) - f(x_2)| < \epsilon$$

is satisfied for any two points x_1, x_2 in $[a, b]$ such that $|x_1 - x_2| < \delta$.

<u>Theorem</u> :- A function which is Continuous in a closed and bounded interval $[a, b]$ is uniformly Continuous in $[a, b]$.

<u>Proof</u>:- By theorem [If f is a Continuous function on the closed interval $[a,b]$, then the interval can always be divided up into a finite number of sub-intervals such that given $\epsilon > 0$

$$|f(x_1) - f(x_2)| < \epsilon$$

where x_1 and x_2 are any two points in the same sub-interval.] the interval $[a, b]$ can be divided up into sub-intervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$$

such that for any two points α, β in the same sub-interval, we have

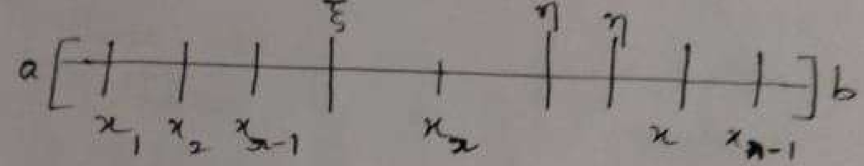
$$|f(\alpha) - f(\beta)| < \tfrac{1}{2}\epsilon \quad \text{——— (1)}$$

Let δ be a positive number which does not exceed the least of the numbers

$$x_1 - a, x_2 - x_1, \ldots, b - x_{n-1}.$$

Let now ξ, η be any two points in $[a, b]$ such that $|\xi - \eta| \not> \delta$.
If these two points are in the same sub-interval, then by 1, we have

$$|f(\xi) - f(\eta)| < \frac{1}{2}\epsilon$$



If ξ, η do not belong to the same sub-interval, then surely they lie one in each of the two consecutive intervals. If x_r is the point of division such that $x_{r-1} < \xi < x_r < \eta < x_{r+1}$, then we have

$$\begin{aligned} |f(\xi) - f(\eta)| &= |f(\xi) - f(x_r) + f(x_r) - f(\eta)| \\ &\leq |f(\xi) - f(x_r)| + |f(x_r) - f(\eta)| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \quad \text{by } 1 \end{aligned}$$

Thus we have shown that, given $\epsilon > 0$ there exists $\delta > 0$

$$|f(\xi - f(\eta)| < \epsilon$$

for any two points ξ, η in $[a, b]$ such that $|\xi - \eta| < \delta$.
Hence f is uniformly continuous in $[a, b]$.