Q.) Prove that $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$

sol:- By De Moivre's theorem, $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

Also, $\sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$

$$\therefore (\sin\theta + i\cos\theta)^4 = \left\{\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right\}^4$$

$$= \cos 4\left(\frac{\pi}{2} - \theta\right) + i\sin 4\left(\frac{\pi}{2} - \theta\right)$$

$$= \cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta) = \cos 4\theta - i\sin 4\theta$$

Hence the given expression $= \frac{\cos 4\theta + i\sin 4\theta}{\cos 4\theta - i\sin 4\theta}$

$$= (\cos 4\theta + i\sin 4\theta)(\cos 4\theta + i\sin 4\theta)$$

$$= (\cos 4\theta + i\sin 4\theta)^2 = \cos 8\theta + i\sin 8\theta.$$

Q.) Prove that $(1+i\sqrt{3})^8 + (1-i\sqrt{3})^8 = -256$

sol:- Put $1 = r\cos\theta$ and $\sqrt{3} = r\sin\theta$

so that $r^2 = 4$ and $\tan\theta = \sqrt{3}$

i.e. $r = 2$ and $\theta = \frac{\pi}{3}$ and we get

$$1 + i\sqrt{3} = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

and $1 - i\sqrt{3} = r\cos\theta - ir\sin\theta = r(\cos\theta - i\sin\theta)$

Therefore, the given expression

$$= \{r(\cos\theta + i\sin\theta)\}^8 + \{r(\cos\theta - i\sin\theta)\}^8$$

$$= r^8(\cos 8\theta + i\sin 8\theta) + r^8(\cos 8\theta - i\sin 8\theta)$$

$$= r^8 \cdot 2\cos 8\theta = (2)^8 \cdot 2\cos 8 \cdot \frac{\pi}{3} = 2^9 \cos\frac{8\pi}{3}$$

$$= 2^9 \cos\left(2\pi + \frac{2\pi}{3}\right) = 2^9 \cos\frac{2\pi}{3}$$

$$= -2^9 \frac{1}{2} = -2^8 = -256$$

(6)

$1^2-(\sqrt{2})^2$
$=1-2=-1$

Q) Prove that $\left(\dfrac{1+\sqrt{2}+i}{1+\sqrt{2}-i}\right)^4 = -1$

Sol:- Let $r\cos\theta = 1+\sqrt{2}$ and $r\sin\theta = 1$ so that

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{1}{1+\sqrt{2}} = \frac{1}{(1+\sqrt{2})} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{1^2-(\sqrt{2})^2}$$

$$= \frac{1-\sqrt{2}}{1-2} = \frac{1-\sqrt{2}}{-1}$$

$$= \sqrt{2}-1$$

(value of $\tan\frac{\pi}{8}$ is $\sqrt{2}-1$) $= \tan\frac{\pi}{8}$

$\therefore \theta = \frac{\pi}{8}$

Now, the given expression $= \left(\dfrac{r\cos\theta + ir\sin\theta}{r\cos\theta - ir\sin\theta}\right)^4$

$$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^4$$

$$= \frac{\cos 4\theta + i\sin 4\theta}{\cos 4\theta - i\sin 4\theta}$$

$$= (\cos 4\theta + i\sin 4\theta)(\cos 4\theta + i\sin 4\theta)$$

$$= (\cos 8\theta + i\sin 8\theta) = \left(\cos 8\cdot\frac{\pi}{8} + i\sin 8\cdot\frac{\pi}{8}\right)$$

$$= \cos\pi + i\sin\pi$$

$= -1 + 0 = -1$ Ans. $\left[\because \theta = \frac{\pi}{8}\right]$

$(\because \cos\pi = -1, \sin\pi = 0)$

Q.) Prove that $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

Sol:- Putting $1+\sin\theta = r\cos\alpha$ and $\cos\theta = r\sin\alpha$

——— (A)

(1)

The given expression

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \left(\frac{r\cos\alpha+ir\sin\alpha}{r\cos\alpha-ir\sin\alpha}\right)^n$$

$$=\left(\frac{\cos\alpha+i\sin\alpha}{\cos\alpha-i\sin\alpha}\right)^n = \frac{\cos n\alpha+i\sin n\alpha}{\cos n\alpha-i\sin\alpha}$$

$$=\frac{\cos n\alpha+i\sin n\alpha}{\cos n\alpha-i\sin n\alpha}$$

$$=\left(\frac{\cos n\alpha+i\sin n\alpha}{\cos n\alpha-i\sin n\alpha}\right)\times\left(\frac{\cos n\alpha+i\sin n\alpha}{\cos n\alpha+i\sin n\alpha}\right)$$

$$=\frac{(\cos n\alpha+i\sin n\alpha)^2}{\cos^2 n\alpha+\sin^2 n\alpha}$$

$$=\cos 2n\alpha+i\sin 2n\alpha \text{ — } (1)$$

But from the given equation (A)

$$\tan\alpha=\frac{\cos\theta}{1+\sin\theta}=\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}$$

$$=\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}} \text{ dividing by } \cos\frac{\theta}{2}+\sin\frac{\theta}{2}$$

$$=\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\therefore \alpha=\frac{\pi}{4}-\frac{\theta}{2}$$

Therefore the given expression (1)

$$=\cos 2n\alpha+i\sin 2n\alpha$$

$$=\cos 2n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin 2n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

8

i.e. 1

Cor 1 Can also be written as
$(\cos 2\alpha + i\sin 2\alpha)^n$

But $2\alpha = 2\cdot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{\pi}{2} - \theta$

Hence $(\cos 2\alpha + i\sin 2\alpha)^n = \left\{\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right\}^n$
$= (\sin\theta + i\cos\theta)^n$.

$\therefore \left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = (\sin\theta + i\cos\theta)^n$

Ex:-7 If $2\cos\alpha = a + \frac{1}{a} \cdot 2\cos\beta$
$= b + \frac{1}{b}$ and $2\cos\gamma = c + \frac{1}{c}$

Show that $2\cos(\alpha+\beta+\gamma) = abc + \frac{1}{abc}$

Sol:- Since $a + \frac{1}{a} = 2\cos\alpha$, therefore we have
$a^2 - 2a\cos\alpha + 1 = 0$
$\therefore a = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$
$= \cos\alpha \pm i\sin\alpha$

Taking the positive sign, $a = \cos\alpha + i\sin\alpha$
Similarly, $b = \cos\beta + i\sin\beta$ and
$c = \cos\gamma + i\sin\gamma$

$\therefore abc = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$ 1

$= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$

Again $\frac{1}{a} = \frac{1}{\cos\alpha + i\sin\alpha} = \cos\alpha - i\sin\alpha$

$\frac{1}{b} = \cos\beta - i\sin\beta$ and
$\frac{1}{c} = \cos\gamma - i\sin\gamma$