Abstract Algebra

Theorem :- If H be a normal subgroup of a group G and K is a normal subgroup of G containing H, then $G/K \cong (G/H)/(K/H)$

Proof :- Since H is a normal subgroup of G and K is a normal subgroup of G containing H. (i.e. $H \subseteq K$)

So the quotient group K/H is a normal subgroup of the quotient group G/H.

Hence: $(G/H)/(K/H)$ is a quotient group.

Now we consider a mapping: $\phi : (G/H) \to (G/K)$ defined by: $\phi(Hx) = Kx$, where $x \in G$.

At first prove: ϕ is well defined.

Let $Hx = Hy$, $x, y \in G$.

$\therefore xy^{-1} \in H \Rightarrow xy^{-1} \in K$ since $H \subseteq K$

$\Rightarrow Kx = Ky \Rightarrow \phi(Hx) = \phi(Hy)$

$\therefore Hx = Hy \Rightarrow \phi(Hx) = \phi(Hy)$. so ϕ is homomorphism.

Let $x, y \in G$.

Then $\phi[(Hx)\cdot(Hy)] = \phi(Hxy) = Kxy$

$= (Kx)(Ky)$

$= \phi(Hx)\phi(Hy)$

so ϕ is homo.

Again: for any $Kx \in G/K$, there must

exists $Hx \in G/H$

such that $\phi(Hx) = Kx$

So ϕ is onto also.

The identity elements of G/K is K.

If $Hx \in G/H$

then $Hx \in$ Kernel of $\phi \Leftrightarrow \phi(Hx) = K$

$\Leftrightarrow Kx = K$

$\Leftrightarrow x \in K$

$\Leftrightarrow Hx \in K/H$

Hence Kernel of $\phi = K/H$, which is a subset of G/H

So ϕ is a homomorphism of G/H onto G/K with Kernel K/H. Therefore by fundamental theorem of homomorphism of groups, we have

$$G/K \cong (G/H)(K/H)$$

Proved

Solvable Groups :-

A group G is said to be solvable if we can find a finite chain of subgroups.

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_k = (e).$$

Such that each N_i is a normal subgroup of N_{i-1} and each quotient group N_{i-1}/N_i is abelian.

Normal series of a group:-

A finite sequence of subgroups $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_k = (e)$ of a group G is called a subnormal series of G if G_{i-1} is a normal subgroup of G_i, $\forall\, i = 0, 1, 2, \ldots, k-1$.

The quotient groups G_i/G_{i+1} are called the factor groups of the subnormal series.

If each G_i is a normal subgroup of G itself, then the series is said to be a normal series of G.

Q.) Prove that every abelian group is solvable.

Sol:- Let G is an abelian group. Let $G = N_0$ and $N_1 = (e)$.

Then $G = N_0 \supseteq N_1 = (e)$ is a solvable series of G.

Clearly $N_1 = (e)$ is a normal subgroup of $N_0 = G$. Since for any $a \in G$ we have

$$aea^{-1} = a^{-1}a = e \in (e) = N_1$$

Since G is abelian, so the quotient group $G/N_1 = G(e) = G$ is abelian.

As every quotient group of an abelian group is abelian. So G is solvable.

Proved