Adjoint Operator - Let $V(F)$ be an inner product space and $T: V \to V$ be linear operator. Let there exists ($\exists$) a unique operator T^* on V such that

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle \quad \forall\ u, v \in V$$

Then T^* is called adjoint of T.

Self-Adjoint Operator - A linear operator T on an inner product space $V(F)$ is called self-adjoint operator iff $T = T^*$ i.e $\langle T(u), v \rangle = \langle u, T(v) \rangle \quad \forall\ u, v \in V$

Unitary operator - A linear operator on an inner product space $V(F)$ is said to be unitary operator if (i) T is one-one onto

(ii) $\langle T(u), T(v) \rangle = \langle u, v \rangle, \ \forall\ u, v \in V$

Def. Let T be a linear transformation from an inner product space $V(F)$ into an inner product space $U(F)$. Then

(i) T preserves norms if $\| T(u) \| = \| u \| \ \forall\ u \in V$

(ii) T preserves inner products if

$$\langle T(u), T(v) \rangle = \langle u, v \rangle \quad \forall\ u, v \in V$$

(iii) T is an isometry if T preserves distances i.e

if $\| T(u) - T(v) \| = \| u - v \| \quad \forall\ u, v \in V$

Def. Let $V(F)$ and $U(F)$ be inner product spaces. Let T be a linear transformation from V into U. Then T is called an inner product space isomorphism if (i) T is invertible i.e T is one-one onto

(ii) T preserves inner product i.e

$$\langle T(u), T(v) \rangle = \langle u, v \rangle \quad \forall\ u, v \in V$$

In this case $V(F)$ and $U(F)$ are called isomorphic.

Imp

Theorem Let T is a linear operator on an inner product space $V(F)$. Then show that adjoint T^* of T exists such that

$TT^* = T^*T = I$ iff T is unitary.

Proof Suppose T is a linear operator on an inner product space $V(F)$ such that

$$TT^* = T^*T = I$$

To prove that T is unitary, then we show that (i) T is invertible

(ii) $\langle T(x), T(y)\rangle = \langle x, y\rangle \quad \forall x, y \in V$

for (i) $\Rightarrow T^*$ exists and $T^*T = I$

$$\Rightarrow T^* = T^{-1}$$

$$\Rightarrow T^{-1} \text{ exists}$$

$$\Rightarrow T \text{ is invertible}$$

for (ii) Let $x, y \in V$ be arbitrary

$$\langle T(x), T(y)\rangle = \langle x, T^*[T(y)]\rangle$$

$$= \langle x, T^*T(y)\rangle$$

$$= \langle x, I(y)\rangle \quad \because TT^* = I$$

$$\langle T(x), T(y)\rangle = \langle x, y\rangle$$

Conversely, suppose that T is a linear operator on an inner product space $V(F)$ s.t. T is unitary.

To prove that T^* exists and $TT^* = T^*T = I$

T is unitary $\Rightarrow T$ is invertible

$\Rightarrow T^{-1}$ exists.

Let $x, y \in V$

$$\langle T(x), y\rangle = \langle T(x), I(y)\rangle$$

$$= \langle T(x), TT^{-1}(y)\rangle \quad \because TT^{-1} = I$$

$$= \langle T(x), T(T^{-1})(y)\rangle$$

$$= \langle x, T^{-1}(y)\rangle$$

For T is unitary $\Rightarrow$ T preserves innerproducts

$$\Rightarrow \langle T(u), T(v)\rangle = \langle u, v\rangle$$

$$\therefore \langle T(x), y\rangle = \langle x, T^{-1}(y)\rangle$$

But $\langle T(x), y\rangle = \langle x, T^{*}(y)\rangle$ by Adjoint op.

$$\therefore \langle x, T^{-1}(y)\rangle = \langle x, T^{*}(y)\rangle$$

This $T^{-1} = T^{*}$

$$\Rightarrow T^{*}T = TT^{*} = I$$