Abstract Algebra

APPLICATIONS OF GALOIS THEORY

The problems which led to the development of the Galois theory consist in the determination of the condition under which:

(i) an equation is solvable by radicals;

(ii) a geometrical construction in a plane is possible by means of ruler and compass only;

(iii) on insolvability of the Quintic.

SOLUTIONS OF POLYNOMIAL EQUATIONS BY RADICALS

Definition 1 :- Let F be a field and an element $a \in F$ be such that it is not the n^{th} power of an element of F, so that $x^n - a \in F[x]$ has not root in F, then a root of $x^n - a$, denoted by $\sqrt[n]{a}$, is called a radical of exponent n over F.

Definition 2 :- A radical $\sqrt[n]{a}$ of exponent n over F is known as reducible or irreducible over F according as $x^n - a$ is reducible or irreducible polynomial of $F[x]$.

Definition 3 :- The equation of the form $x^n - a = 0$ is known as pure equation.

An extension field $F(\sqrt[n]{a})$ is known as pure extension of F.

Definition 4 :- A field extension which can be reached through a finite series of successive pure extensions, is referred to as solvable by radicals or as a Tower.

If K be a given field F we have

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_r = K$$

Such that each member of the series is a pure extension of its predecessor, then K is referred to as a Tower over F or as a solvable by radicals over F.

Solution by Radicals

Let F be a field and $f(x) \in F[x]$ be a non-constant polynomial. In this section we shall consider the problem of solving the equation $f(x) = 0$.

If $f(x) = x^2 + bx + c$ over a field F of characteristic $\neq 2$ then the solutions of $f(x) = 0$ are given by $-b \pm \sqrt{b^2 - 4c}$, so that this equation either has solutions in F or solutions in $F(\sqrt{b^2-4c})$. Thus $f(x) = 0$ can always be solved in some field which is obtained by adjoining a radical to F. We wish to determine when an arbitrary polynomial equation $f(x) = 0$ can be solved, if it is not solved in F, then there exists some field in which $f(x) = 0$ will have the solution, this field is obtained from F by the successive adjunction of radicals.

Definition 5:- Let $f(x) \in F[x]$ be a polynomial then the polynomial equation $f(x) = 0$ is said to be solvable by radicals over F, if the splitting field K of $f(x)$ is a tower over F.

If K is a normal extension of F, then the tower is said to be normal radical tower over F.