Set up Schrodinger wave equation for a rigid rotator with free axis and solve it to obtain energy eigen values.

Rigid Rotator:-

A rigid rotator is considered to be dumbell shaped and considering of two small masses m_1 and m_2 connected together, separated by a finite fixed distance. The system is capable of rotation about an axis passing through the centre of mass of the system and perpendicular to the line joining two masses. If the direction of the axis of rotation remains fixed in space, the system is called a rigid rotator with fixed axis. If the plane of these two masses can move, then the axis of rotation is free to take any position in space and hence the system is called the rigid rotator with free axis.

Rigid Rotator with Free Axis:-

Let us suppose that the co-ordinates of two masses with respect to the origin O be (x_1, y_1, z_1) and (x_2, y_2, z_2). If the distance of mass m_1 from O be r_1 and of mass m_2 be r_2, then their polar co-ordinates can be expressed as (r_1, θ, ϕ) and $(r_2, \theta+\pi, \phi+\pi)$, as shown in the figure.

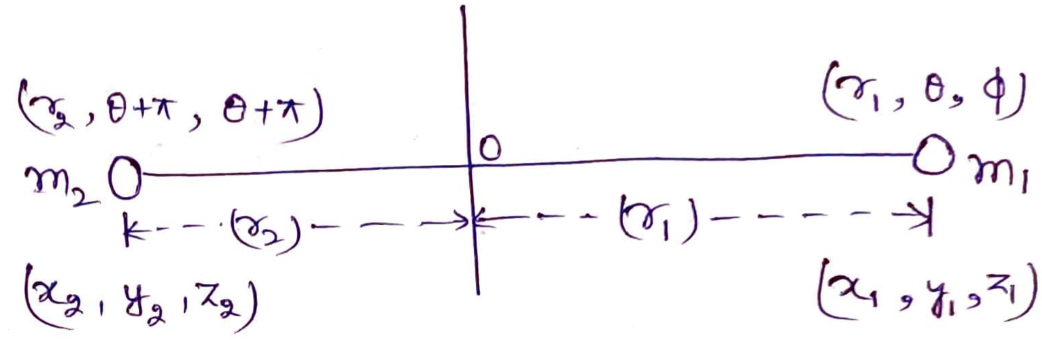

Fig - 1

The cartesian co-ordinates are related to the respective polar co-ordinates by the relations,

$$x_1 = r_1 \sin\theta \cos\phi, \qquad x_2 = r_2 \sin(\theta+\pi)\cos(\phi+\pi)$$

$$= r_2 \sin\theta \cos\phi$$

$$y_1 = r_1 \sin\theta \sin\phi,$$

$$z_1 = r_1 \cos\theta, \qquad y_2 = r_2 \sin(\theta+\pi)\sin(\phi+\pi)$$

$$= r_2 \sin\theta \sin\phi$$

$$z_2 = r_2 \cos(\theta+\pi)$$

$$= -r_2 \cos\theta.$$

Now,
$$\frac{dx_1}{dt} = r_1\left[\cos\theta\cos\phi\left(\frac{d\theta}{dt}\right) - \sin\theta\sin\phi\left(\frac{d\phi}{dt}\right)\right]$$

$$\frac{dy_1}{dt} = r_1\left[\cos\theta\sin\phi\left(\frac{d\theta}{dt}\right) + \sin\theta\cos\phi\left(\frac{d\phi}{dt}\right)\right]$$

$$\frac{dz_1}{dt} = -r_1 \sin\theta\left(\frac{d\theta}{dt}\right)$$

$$\therefore \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2$$

$$= r_1^2\left[\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2 + \sin^2\theta\left(\frac{d\theta}{dt}\right)^2\right]$$

$$= r_1^2\left[\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]$$

Similarly,

$$\left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2 = r_2^2\left[\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]$$

In such a system the energy is only kinetic energy of rotation.

$$T = T_1 + T_2 = \frac{1}{2}m_1 r_1^2\left[\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] + \frac{1}{2}m_2 r_2^2\left[\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]$$

$$= \left(\frac{1}{2}m_1 r_1^2 + \frac{1}{2}m_2 r_2^2\right)\left[\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] \qquad (1)$$

Here, $m_1 r_1^2 + m_2 r_2^2 = I$, the moment of inertia of the system. Thus equation (1) reduces to

$$T = \frac{I}{2}\left[\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] \qquad (2)$$

This kinetic energy is the same as that of a single particle of mass I moving on the surface of a sphere of unit radius.