Derive Lagrange's equation of motion for holonomic system in terms of generalized co-ordinates.

In Langrangian equation energy plays a prime role. Let the position vector of the i th particle of mass m_i of a holonomic system at the instant 't' be r_i, then the Kinetic energy of the system is given by

$$T = \frac{1}{2} \sum_i m \dot{r}_i^2 \qquad \text{——— (1)}$$

Where,

$$r_i = r_i(q_1, q_2, \dots, q_n, t). \qquad \text{——— (2)}$$

We know by differentiating equation (2) with respect to 't';

$$\dot{r}_i = \sum_i \frac{\partial r_i}{\partial q_K} \dot{q}_K + \frac{\partial r}{\partial q_K} q_K + \frac{\partial r_i}{\partial t} \text{; hence}$$

From eq^n (1);

$$T = \frac{1}{2} \sum m_i \left(\sum_i \frac{\partial r_i}{\partial q_K} \dot{q}_K + \frac{\partial r_i}{\partial t} \right)^2$$

$$= \frac{1}{2} \sum_i m_i \left[\sum_{K,i} \frac{\partial r_i}{\partial q_K} \cdot \frac{\partial r_i}{\partial q_l} \dot{q}_K \dot{q}_l + 2 \sum_K \frac{\partial r_i}{\partial q_K} \cdot \frac{\partial r_i}{\partial t} \dot{q}_K + \left(\frac{\partial r_i}{\partial t} \right)^2 \right]$$

$$= \sum_{i,K,l} \frac{1}{2} m_i \frac{\partial r_i}{\partial q_K} \cdot \frac{\partial r_i}{\partial q_l} \dot{q}_K \dot{q}_l + \sum_{i,K} m_i \frac{\partial r_i}{\partial q_K} \cdot \frac{\partial r_i}{\partial t} \dot{q}_K + \sum \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t} \right)^2$$

$$= \sum_{k,l} a_{kl} \dot{q}_k \dot{q}_l + \sum_k a_k \dot{q}_k + a \quad \text{--- (3)}$$

where

$$\left.\begin{aligned} a_{kl} &= \sum_i \frac{1}{2} m_i \frac{\partial r_i}{\partial q_k} \cdot \frac{\partial r_i}{\partial q_l} \\ a_k &= \sum_i m_i \frac{\partial r_i}{\partial q_k} \cdot \frac{\partial r_i}{\partial t} \\ a &= \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t}\right)^2 \end{aligned}\right\} \quad \text{--- (4)}$$

The coefficients a_{kl}, a_k and a in eqn (3) are functions of r and t.

From eqn. (3), it is seen that kinetic energy is a quadratic function of the generalized velocities.

A case of considerable importance arises when t is not explicitly involved. Then $(\partial r/\partial t) = 0$ and therefore $a_k = 0$ and $a = 0$, so eqn. (3) reduces to

$$T = \sum_{k,l} a_{kl} q_k q_l \quad \text{--- (5)}$$

Since dot product commutes, eqn. (4) shows that $q_{kl} = q_{lk}$ and thus equation (5) states that kinetic energy (K.E) is a homogeneous quadratic function of the generalised velocities.

Using Euler's theorem for homogeneous functions, we get.

$$2T = \dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + \cdots \dot{q}_n \frac{\partial T}{\partial \dot{q}_n}$$

or, $$2T = \sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = \sum_{j=1}^{n} p_j \dot{q}_j \quad \text{——— (6)}$$

Where $\frac{\partial T}{\partial \dot{q}_j} = p_j$ = generalised momenta.

Now the Langrangian equation in generalised coordinate q_k is

$$\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \left(\frac{\partial T}{\partial q_k}\right) = Q_k \Rightarrow \left(\frac{d}{dt}\right)\left(\frac{\partial T}{\partial \dot{x}}\right) - \left(\frac{\partial T}{\partial x}\right) = F_x;$$

$$\left(\frac{d}{dt}\right)\left(\frac{\partial T}{\partial \dot{y}}\right) - \left(\frac{\partial T}{\partial y}\right) = F_y;$$

$$\left.\left(\frac{d}{dt}\right)\left(\frac{\partial T}{\partial \dot{z}}\right) - \left(\frac{\partial T}{\partial z}\right) = F_z \right\} \text{——— (7)}$$

Where, F_x, F_y, F_z are the generalised forces along x, y and z axes respectively.

From eqn. (7), $\left(\frac{d}{dt}\right)(m\dot{x}) = F_x$

i.e; $m\ddot{x} = F_x$ and similarly.

$m\ddot{y} = F_y$ and $m\ddot{z} = F_z$,

or, in vector form, $m\ddot{r} = F$, where $r = (x, y, z)$ and $F = (X, Y, Z)$. ——— (8).