Exp. Prove that the series

$$1+\frac{\alpha}{\beta}+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}+\cdots\cdots$$

where $(\alpha>0,\ \beta>0)$

Converges if $\beta>\alpha+1$ and diverges if $\beta\le\alpha+1$.

Solution:- Given series

$$1+\frac{\alpha}{\beta}+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}+\cdots$$

on ignoring the first term of the given series then series is positive term series and nth term of the positive term series is

$$a_n=\frac{\alpha(\alpha+1)(\alpha+2)+\cdots\cdots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)+\cdots\cdots(\beta+n-1)}$$

$$a_{n+1}=\frac{\alpha(\alpha+1)(\alpha+2)+\cdots\cdots(\alpha+n-1)(\alpha+n)}{\beta(\beta+1)(\beta+2)+\cdots\cdots(\beta+n-1)(\beta+n)}$$

$$\because \lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\left(\frac{\beta+n}{\alpha+n}\right)$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{(1+\beta/n)}{(1+\alpha/n)}=1$$

Hence Ratio Test fails, we apply Raabe's Test

Here $\dfrac{a_n}{a_{n+1}} = \dfrac{(\beta+n)}{(\alpha+n)}$

Then $n\left(\dfrac{a_n}{a_{n+1}} - 1\right) = n\left[\dfrac{(\beta+n)}{(\alpha+n)} - 1\right]$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left[\frac{\beta-\alpha}{\alpha+n}\right]$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n\,(\beta-\alpha)}{n\,(1+\alpha/n)}$$

$$= \lim_{n\to\infty} \frac{(\beta-\alpha)}{(1+\alpha/n)}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = (\beta-\alpha)$$

By Raabe's Test the series converges if $\beta-\alpha > 1$ i.e $\beta > \alpha+1$ and

diverges if $\beta-\alpha < 1$ i.e $\beta < \alpha+1$.

Raabe's Test fails if $\beta-\alpha = 1$ i.e $\beta = \alpha+1$

then we obtain a_n for $\beta = \alpha+1$ putting in

$$a_n = \frac{\alpha\,\cancel{(\alpha+1)}\,\cancel{(\alpha+2)}\cdots\cdots\cancel{(\alpha+n-1)}}{\cancel{(\alpha+1)}\,\cancel{(\alpha+2)}\cdots\cdots\cancel{(\alpha+n-1)}\,(\alpha+n)} = \frac{\alpha}{\alpha+n}$$

$\because a_n = \frac{\alpha}{\alpha+n} \simeq \frac{1}{n}$

Let $b_n = \frac{1}{n}$, so that $\sum b_n$ diverges

Now $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\alpha}{(\alpha+n)} \cdot \frac{n}{1}$

$$= \lim_{n\to\infty} \frac{\alpha}{1+\alpha/n} = \alpha > 0$$

So. $\sum a_n$ diverges, since $\sum b_n$ diverges.

Hence, the given series converges for $\beta > \alpha+1$ and diverges for $\beta \leq \alpha+1$.

Que.:- Test for converges the series

$$1 + \alpha + \frac{\alpha(\alpha+1)}{1\cdot 2} + \frac{\alpha(\alpha+1)(\alpha+2)}{1\cdot 2\cdot 3} + \cdots$$

Sol. Solution same as above Example.

$\because$ Hint $a_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{1\cdot 2\cdot 3\cdots n}$

$a_{n+1} = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(\alpha+n)}{1\cdot 2\cdot 3\cdots n\,(n+1)}$

Apply Ratio & Raabe's tests for solution.