For T.D.C. part(II) in phy (H & Sub)

Topic:- Energy Associated with Electromagnetic Field (i.e Poynting theorem)

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When an electromagnetic field is traveling in a space, it transports energy with it. In order to explain it, consider a volum V bounded by a closed surface S. Let the material inside S be isotropic, homogeneous and characterized by permeability μ, permittivity ϵ and conductivity σ. The electric field $\vec{E}$ and magnetic field $\vec{B}$ of the em wave in this vol^m are related by Maxwell's third & fourth equations as

$$\text{Curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \cdots\cdots (1)$$

$$\& \quad \text{Curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \cdots (2)$$

Taking scalar product on both sides of eq^n (1) with $\vec{H}$ and on both sides of eq^n (2) by $\vec{E}$, we have

$$\vec{H}\cdot \text{Curl}\,\vec{E} = -\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} \rightarrow (3)$$

$$\& \quad \vec{E}\cdot \text{Curl}\,\vec{H} = \vec{J}\cdot\vec{E} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \rightarrow (4)$$

Subtracting eq^n (3) from equation (4), we have

$$\vec{E}\cdot \text{Curl}\,\vec{H} - \vec{H}\cdot \text{Curl}\,\vec{E} = \vec{J}\cdot\vec{E} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t} \rightarrow (5)$$

Now, as $\operatorname{div}(\vec{E}\times\vec{H}) = \vec{H}\cdot \operatorname{Curl}\vec{E} - \vec{E}\cdot \operatorname{Curl}\vec{H}$

~~And~~, $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$

$\therefore$ Eq^n (5) becomes,

$$-\operatorname{div}(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \left[\vec{E}\cdot\frac{\partial}{\partial t}(\epsilon\vec{E}) + \vec{H}\cdot\frac{\partial}{\partial t}(\mu\vec{H})\right]$$

$$= \vec{J}\cdot\vec{E} + \left[\epsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \mu\,\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}\right] \longrightarrow (6)$$

Again as $E^2 = \vec{E}\cdot\vec{E}$

$$\therefore 2E\frac{dE}{dt} = \vec{E}\cdot\frac{d\vec{E}}{dt} + \frac{d\vec{E}}{dt}\cdot\vec{E} = 2\vec{E}\cdot\frac{d\vec{E}}{dt}$$

or, $$\frac{d}{dt}E^2 = 2\vec{E}\cdot\frac{d\vec{E}}{dt}$$

$$\therefore \vec{E}\cdot\frac{d\vec{E}}{dt} = \frac{1}{2}\frac{dE^2}{dt}$$

& similarly, $\vec{H}\cdot\frac{d\vec{H}}{dt} = \frac{1}{2}\frac{dH^2}{dt}$

$\therefore$ Eq^n (6) becomes

$$-\operatorname{div}(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)\right] \longrightarrow (7)$$

Integrating volum V bounded by the surface S, we get

$$-\int_V \operatorname{div}(\vec{E}\times\vec{H})\,dV = \int_V (\vec{J}\cdot\vec{E})\,dV + \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)dV \longrightarrow (8)$$

Let us interpret the terms of ~~left~~ eq^n (8).

$\Rightarrow$ Interpretation of $\int_V (\vec{J}\cdot\vec{E})\,dV \Longrightarrow$:

Here, $$\int_V (\vec{J}\cdot\vec{E})\,dV = \int_V (\vec{J}\,dv\cdot\vec{E})$$

$$= \int_V (\vec{J}\,ds\,dl)\cdot\vec{E}$$

$$= \int_V (\vec{J}\,ds)\,dl\cdot\vec{E}$$

$$= \int_V I\vec{dl} \cdot \vec{E}$$

$$= \int_V \frac{dq}{dt} dl \cdot \vec{E}$$

$$= \int_V dq\, v \cdot \vec{E}$$

$$= \sum_j q_j (v_j \cdot \vec{E}_j) \quad \ldots\ldots (9)$$

Where $\vec{E}_j$ denotes the electric field at the position of charge q_j. The force experienced by the point charge is equal to the ~~electric~~ sum of electric force and magnetic force

i.e $F = q(\vec{E} + \vec{v} \times \vec{B})$

$\therefore$ Work done by the force in displacing the charge by an infinitesimal distance dl will be

$$dW = \vec{F} \cdot \vec{dl}$$

$$= q\vec{E} \cdot \vec{dl} + q(\vec{v} \times \vec{B}) \cdot \vec{dl}$$

$$= q\vec{E} \cdot \vec{dl} + q\left(\frac{\vec{dl}}{dt} \times \vec{B}\right) \cdot \vec{dl}$$

$$= q\vec{E} \cdot \vec{dl} + q\left(\frac{\vec{dl}}{dt} \cdot \vec{dl}\right) \times \vec{B}$$

$$= q\vec{E} \cdot \vec{dl} \quad \left(\because \frac{\vec{dl} \times \vec{dl}}{dt} = 0\right)$$

$$\therefore \frac{dW}{dt} = q\vec{E} \cdot \vec{v}$$

Including all charges, we have

$$\frac{dW}{dt} = \sum_j q_j \vec{E}_j \cdot \vec{v}_j \longrightarrow (10)$$

From equation (9) and (10), we have

$$\int_V (\vec{J}\cdot\vec{E})\,dV = \frac{dW}{dt} \rightarrow (11)$$

which represent the rate at which work is done by the field on the charges.

⟹ Interpretation of $\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dV$:—

In this integration the first & 2nd term represent the time rate of increase of energy stored in the electric and magnetic fields respectively in the volume V.

Thus, from the above interpretation we conclude that R.H.S of eq^n (8) represents the sum of the power expended by the fields due to the motion of charge and the time rate of increase of stored energy in the fields. Obviously, the L.H.S of eq^n (8) must represent the power flow into the volume V across the surface S or the power flow out of volume V across the surface S,

$$= \int_V \mathrm{div}(\vec{E}\times\vec{H})\,dV$$

$$= \oint_S (\vec{E}\times\vec{H})\cdot\vec{dS}$$

$$= \oint_S \vec{P}\cdot\vec{dS}$$

where $\vec{P} = \vec{E}\times\vec{H} \longrightarrow (12)$ & is known as "poynting vector." Thus $\mathrm{div}(\vec{E}\times\vec{H})$ represent the time rate of electromagnetic energy flowing into or out of the vol^m V across surface S due to flow of electromagnetic wave & is known as Poynting Theorem.