PG SEMISTR II, UNIT III, SYMMETRY ELEMENTS

BY

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Chapter 3 - Group Theory

A Group is a collection of elements which is:

i) **closed under some single-valued associative binary operation**

ii) **contains a single element satisfying the identity law**

iii) **and has a reciprocal element for each element in the group**

Collection: a specified # of elements (finite or infinite)

Elements: the consitituents of the group (i.e., symmetry operations)

Binary Operation: the combination of two elements of a group to yield another element in the group. The combination may be mathematical (addition, subtraction, etc.) or qualitative as in the successive application of two symmetry operations on an object.

Single-valued: the combination of two elements yields a unique result

Closed: the combination of any two group elements must always yield another element belonging to the group.

Associative: the associative law of combination must hold for the group.

$$(AB)C = A(BC)$$

In general, however, elements of a group do not have to commute (but they can):

$$AB \neq BA$$

Identity Law: there must be an element in the group which when combined with any element in the group will leave them unchanged. This element is called the identity or unit element and it commutes with all elements of the group. It is given the symbol E.

$$EA = A \qquad AE = A \qquad EE = E$$

Reciprocal Element: for each element A in a group there must be an element called the reciprocal, A^{-1}, such that the following holds:

$$AA^{-1} = A^{-1}A = E$$

In general, group multiplication is not commutative, i.e., AB ≠ BA. However, it can be and a group in which multiplication is completely commutative is called an Abelian Group.

Group Multiplication Table (matrix operations)

G_3	E	A	B
E	E	A	B
A	A	B	E
B	B	E	A

(column) × (row)

Each row and each column in a group multiplication table lists each of the group elements ONCE and ONLY ONCE. It therefore follows that no two columns or rows may be identical!

Consider a "real" C_3 table using symmetry elements:

C_3	E	C_3	C_3^2
E	E	C_3	C_3^2
C_3	C_3	C_3^2	E
C_3^2	C_3^2	E	C_3

Abelian group

Consider the two different ways we can set up a 4 × 4 table:

G_4^1	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

Note that each element times itself generates E.

G_4^2	E	A	B	C
E	E	A	B	C
A	A	B	C	E
B	B	C	E	A
C	C	E	A	B

Note that this group table above is cyclic, that is, the group is generated by one element:

$$A = A \qquad A^3 = C$$

$$A^2 = B \qquad A^4 = E$$

Note that the G_3 (C_3) example above was also cyclic.

There is only one group combination possible for the G_5 group, which turns out to be cyclic as well:

G_5	E	A	B	C	D
E	E	A	B	C	D
A	A	B	C	D	E
B	B	C	D	E	A
C	C	D	E	A	B
D	D	E	A	B	C

Note the diagonal lining up of the elements in cyclic groups (symmetry of a matrix sort).