Finite Difference

A finite difference is a mathematical expression of the form: $f(x+b) - f(x+a)$.

If a finite difference is divided by $b-a$, one gets a difference quotient. i.e. $\frac{\Delta f(x)}{h} = \frac{f(x+b) - f(x+a)}{b-a}$

Finite difference is often used as an approximation of the derivative, typically in numerical differentiation.

$$f'(x) = \lim_{(b-a)\to 0} \frac{[f(x+b) - f(x+a)]}{(b-a)}$$

or. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \because h = (b-a)$$

If h has a fixed (non-zero) value instead of approaching zero (0), then right-hand side of above Equation as-

$$\frac{f(x+h) - f(x)}{h} = \frac{\Delta f(x)}{h}$$

So, $\Delta f(x)$ is finite difference divided by equally interval h. Hence, the constant difference between two consecutive points of x is called the interval of differencing denoted by h.

Finite Differences-

Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \ldots n$. of any function $y = f(x)$, the values of x being equally spaced (interval), i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots n$.

Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x in the range $x_0 \leq x \leq x_n$. or

To obtain the derivative of $f(x)$ for some x in range $x_0 \leq x \leq x_n$.

The methods for the solution to these problems are based on the concept of the "differences" of a function.

Finite difference is often used as an approximation of the derivative in numerical differentiation.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If h as a fixed (non-zero) value instead of approaching zero, then right hand side of the above equation as-

$$\frac{f(x+h) - f(x)}{h} = \frac{\Delta f(x)}{h}$$

So, $\Delta f(x)$ is finite difference divided by equally interval h.

The following differentce operators are used for find the differences.

* Forward difference operator (Δ):-

Suppose that a function $y = f(x)$ is given at equally space points as $x_0, x_1, x_2, \dots$ as $y_0, y_1, y_2 \dots y_n$ respectively. Also let the constant difference between two consecuti points of x is called the interval of differ-cing h. Then the forward difference ope Δ is defined as.

$$\Delta f(x) = f(x+h) - f(x)$$

or $\Delta y_i = y_{i+1} - y_i$, $i = 0, 1, 2 \dots$

So $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, $\dots$ $\Delta y_n = y_{n+1} - y_n$

and $x_i = x_0 + ih$

Similarly the higher differences are defin

$$\Delta^2 y_i = \Delta(\Delta y_i)$$

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+h) - \Delta f(x))$$

or in general

$$\Delta^n f(x) = \Delta^{n-1}(\Delta f(x))$$

$$\Delta^n f(x) = \Delta^{n-1}(f(x+h) - f(x)$$

$$\Delta^n f(x) = \Delta^{n-1}(f(x+h)) - \Delta^{n-1} f($$

⇒ Let a function:-

$$y = f(x), \quad x_0 \le x \le x_n$$

have a table of values (x_i, y_i), $i = 0, 1, 2 \cdots n$. These values of x are equally spaced (interval) i.e. $h = 1$.

$$x_i = x_0 + ih, \quad i = 0, 1, 2 \cdots n.$$

$$x_0 = x_0 + 0 \Rightarrow y_0 = f(x_0)$$

$$x_1 = x_0 + h \Rightarrow y_1 = f(x_0 + h)$$

$$x_2 = x_0 + 2h \Rightarrow y_2 = f(x_0 + 2h)$$

$$\vdots$$

$$x_n = x_0 + nh \Rightarrow y_n = f(x_0 + nh)$$

If $y_0, y_1, y_2, \cdots y_n$ denote a set of values of y, then $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, $\cdots$ $y_n - y_{n-1}$ are called the differences of y.

These differences denote as:-

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\Delta y_2 = y_3 - y_2$$

$$\vdots$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called the forward difference

- differences. The differences of the first forward difference are called second forward differences and are denoted by

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 \Rightarrow (y_2 - y_1) - (y_1 - y_0)$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 \Rightarrow (y_3 - y_2) - (y_2 - y_1)$

$\Delta^2 y_2 = \Delta y_3 - \Delta y_2 \Rightarrow (y_4 - y_3) - (y_3 - y_2)$

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$\Delta^2 y_{n-1} = \Delta y_{n-1} - \Delta y_{n-2} \Rightarrow (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$

Similarly we can define third forward differences, fourth forward differences etc.. Thus,

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$

$= y_3 - 3y_2 + 3y_1 - y_0$

$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) -$

$(y_3 - 3y_2 + 3y_1 - y_0)$

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Exp. If $y = f(x)$ is known at these points.

x_i	0	1	2	3	4
y_i	1	7	23	55	109

To find the forward difference Table.

Sol.

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				