* Central difference interpolation formulae –

As we know, Newton's forward and backward interpolation formulae are used to interpolation value of x, $(x_0 \le x \le x_n)$ for given set values at near the beginning and ending of the table set differences, respectively.

* Thus, the central difference formulae are most suitable for interpolation near the middle of a tabulated set. The central difference operator δ was already discussed in previous lecture.

The most important central difference formulae are Gauss, Stirling, Bessel and Everett. So, Gauss's formulae discussed below are of interest from a theoretical stand-point only.

Gauss's Forward formula – We consider, the central ordinate (y_0) is taken for convenience according to $x = x_0$.

The differences used in the formula lie on the line shown in Gauss's Forward table.

The formula is of the form

$$y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} \ldots \quad (1)$$

where $G_1, G_2, \ldots$ have to be determined.
The y_p of the left side can be expressed in terms of $y_0, \Delta y_0$ and higher-order differences of y_0.

Table — Gauss's forward Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$ Back.	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$ forw.	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

So, clearly

$$y_p = E^p y_0 \qquad E = (1+\Delta) \Rightarrow \text{shift oper.}$$

$$y_p = (1+\Delta)^p y_0$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \cdots$$

Similarly, the righ side of Eqⁿ (7) can be expressed in terms of y_0, Δy_0 and higher-order differences.

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0$$

We know

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0$$
$$= \Delta^2 (1+\Delta)^{-1} y_0$$
$$= \Delta^2 (1-\Delta+\Delta^2-\Delta^3+\cdots) y_0$$
$$= \Delta^2 (y_0 - \Delta y_0 + \Delta^2 y_0 - \Delta^3 y_0 + \cdots)$$
$$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots$$

Then $\Delta^3 y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \Delta^6 y_0 + \cdots$

$$\Delta^4 y_{-2} = \Delta^4 E^{-2} y_0$$
$$= \Delta^4 (1+\Delta)^{-2} y_0$$
$$= \Delta^4 (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \cdots) y_0$$
$$= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots$$

Hence, Eqn (1) gives the identity

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \cdots$$
$$= y_0 + G_1 \Delta y_0 + G_2 (\Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \cdots)$$
$$+ G_3 (\Delta^3 y_0 - \Delta^4 y_0 + \Delta^5 y_0 - \cdots)$$
$$+ G_4 (\Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 - 4\Delta^7 y_0 + \cdots) \quad (11)$$
$$+ \cdots$$

Equating the coefficients of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ etc. on both side in Eqn (11)

$$G_1 = p, \quad G_2 = \frac{p(p-1)}{\lfloor 2}, \quad G_3 = \frac{(p+1)p(p-1)}{\lfloor 3},$$
$$G_4 = \frac{(p+1)p(p-1)(p-2)}{\lfloor 4}, \ldots$$

Similarly for–

Gauss's Backward Formula → This formula used the differences ~~with~~ which lie on the line (dashed) shown in table.

Gauss's backward formula can be assumed to be of the form.

$$y_p = y_0 + G_1' \Delta y_{-1} + G_2' \Delta^2 y_{-1} + G_3' \Delta^3 y_{-2} + G_4' \Delta^4 y_{-2}$$

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where G_1', G_2', G_3' -- have to be determined. following the same procedure as in Gauss's forward formula, we obtain–

$$G_1' = p, \quad G_2' = \frac{p(p+1)}{\lfloor 2}, \quad G_3' = \frac{p(p+1)(p-1)}{\lfloor 3}$$

$$G_4' = \frac{(p+2)(p+1)p(p-1)}{\lfloor 4}, \dots$$

Stirling's formula – Taking mean of Gauss's forward and backward formulae, we obtain

$$y_p \doteq y_0 + p\,\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2}\,\Delta^2 y_{-1} + \frac{p(p^2-1)}{\lfloor 3}\,\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}$$

$$+ \frac{p^2(p^2-1)}{\lfloor 4}\,\Delta^4 y_{-2} + \cdots$$

This is called Stirling's formula.