*: Interpolation is the process/technique of estimating the value of a function for any intermediate value of the independent variable.

Newton's Forward Difference Interpolation formula:-

Given the set of values, viz. (x_0, y_0), (x_1, y_1), (x_2, y_2) ...... (x_n, y_n) of x and y. it is required to find $y_n(x)$, a polynomial of nth degree such that y and $y_n(x)$ agree at the table points.

Let $x_i = x_0 + ih$ $(i = 0, 1, 2, \ldots n)$

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_n)$$

y and $y_n(x)$ should agree at the set of table points, we obtain.

$$a_0 = y_0, \quad a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h},$$

$$a_2 = \frac{\Delta^2 y_0}{\underline{|2}\, h^2}, \quad a_3 = \frac{\Delta^3 y_0}{\underline{|3}\, h^3}, \ldots$$

$$a_n = \frac{\Delta^n y_0}{\underline{|n}\, h^n}$$

Setting $x = x_0 + ph$ and substituting $a_0, a_1, a_2, \ldots a_n$ in $y_n(x)$ equation. then gives and $p = (x - x_0)/h$.

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\underline{|2}}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\underline{|3}}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\cdots(p-n)}{\underline{|n}}\cdot\Delta^n y_0$$

Which is Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabular values.

Exp. Find $f(x)$ at the following data points.

x_i	0	1	2	3	4
$f_i(x)$	1	7	23	55	109

then find $f(0.5)$ and $f(1.5)$ using Newton's forward difference formula.

Sol. Forward difference table

x_i	$f_i(x)=y_i$	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_0$
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				

Here $h=1$, $x_0=0$ then $x = x_0 + ph$

$x = 0 + p\cdot 1 \Rightarrow x = p$

using Newton's forward difference formula

NFDI formula -

$$f(x) = y_0 + \frac{p \Delta y_0}{\lfloor 1} + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$
$$+ \frac{p(p-1)(p-2)(p-3)}{\lfloor 4} \cdot \Delta^4 y_0 \quad \text{--- (i)}$$

Substituting values of p and $y_0, \Delta y_0, \Delta^2 y_0 \ldots$ in this equation (i)

$$f(x) = 1 + \frac{x \cdot 6}{\lfloor 1} + \frac{x \cdot (x-1)}{\lfloor 2} \cdot 10 + \frac{x(x-1)(x-2)}{\lfloor 3} \cdot 6$$
$$+ \frac{x(x-1)(x-2)(x-3)}{\lfloor 4} \cdot 0$$

$$f(x) = 1 + 6x + 5(x^2 - x) + (x^2 - x)(x-2)$$
$$= 1 + 6x + 5x^2 - 5x + x^3 - x^2 - 2x^2 + 2x$$
$$f(x) = x^3 + 2x^2 + 3x + 1 \quad \text{--- (ii)}$$

function is a third degree polynomial and hence third forward differences are constant

put $x = 0.5$

$$f(0.5) = (0.5)^3 + 2(0.5)^2 + 3(0.5) + 1$$
$$= .125 + 0.50 + 1.5 + 1$$
$$f(0.5) = 3.125$$

Similarly putting $x = 1.5$ in Eqn (ii) then we get

$$f(1.5) = (1.5)^3 + 2(1.5)^2 + 3(1.5) + 1$$
$$= 3.375 + 4.5 + 4.5 + 1$$
$$f(1.5) = 13.375$$

Newton's Backward Difference Interpolation

Instead of assuming $y_n(x)$ is a polynomial of the nth degree function for Newton's forward difference interpolation, if we choose it in the form-

$$y_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots + a_n(x-x_n)(x-x_{n-1})\cdots(x-x_1).$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, x_{n-2}, \cdots x_1, x_0$. Then we obtain-

$$a_0 = y_n; \quad a_1 = \frac{y_n - y_{n-1}}{x - x_n} = \frac{\nabla y_n}{h}; \quad a_2 = \frac{\nabla^2 y_n}{\lfloor 2 \cdot h^2}$$

here $p = \frac{x-x_n}{h} \Rightarrow x - x_n = ph.$

Then

$$y_n(x) = y_n + \frac{p \nabla y_n}{\lfloor 1} + \frac{p(p+1)}{\lfloor 2} \nabla^2 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{\lfloor n} \nabla^n y_n$$

This is Newton's Backward difference interpolation formula and it uses tabular values to the left of y_n.