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Variational or Hamilton's Principle

Derive Lagrangian equation of Motion from this Principle

A more general formulation of the Lagrangean in mechanics is due to the variational principle (so called Hamilton's principle). The principle is stated in a generalised form independent of any co-ordinates system and hence is useful in non-mechanical systems and fields also. It is also known as integral principle.

This principle may be stated as — "out of all the possible paths along which a dynamical system may move from one point to another point within a given interval of time (consistent with constraints if any), the actual path followed is that which minimizes the time integral of the Lagrangian."

Analytically it can be represented as

$$I = \int_{t_1}^{t_2} L\,dt = \text{extremum} \qquad \text{—— (1)}$$

where I is the extremum value of time integral of Lagrangean and is known as Hamilton's principal function of the path.

Taking δ-variation of the eqn. (1), the variational principle may also be represented as

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \qquad \text{—— (2)}$$

where $L = T - V$, $T = K.E$ & $V = P.E.$

In order to deduce Lagrangean Eqn. of Motion from this Principle, let us take a conservative system of particles for which Lagrangian is given by

$$L = L(q_k, \dot{q}_k, t) \qquad \text{—— (3)}$$

But due to homogeneity of time, the Lagrangian for a dynamical system is not an explicit fn of time. Thus, the eqn (3) reduces to

$$L = L(q_k, \dot{q}_k) \qquad \text{—— (4)}$$

~~Now using~~ Hence $$\delta L = \sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k \qquad \text{—— (5)}$$

Now using eqn. (5) in eqn. (2), we obtain

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}(\delta q_k)\, dt = 0 \quad \text{Since } \delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$$

or $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \sum_k \left[\left\{ \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right\}_{t_1}^{t_2} - \int_{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k\, dt \right] = 0$$

(Integrating by Parts —— (6)

But $\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$ because $\delta q_k = 0$ at end points

Therefore eqn. (6) reduces to

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt - \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k\, dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right] \delta q_k\, dt = 0 \quad \text{—— (7)}$$

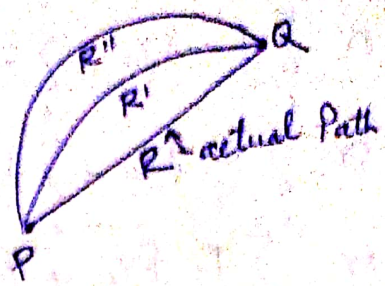
But variables being independent, the variations δq_k independent if and only if

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

i.e. $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

Which is Langrange's equation of Motion

Deduction of Hamilton's Principle



Let us consider that the conservative holonomic dynamical system moves from P to Q where P & Q are initial and final points at time t_1 & t_2 respectively. Let PRQ be the actual path and PR'Q & PR''Q be the two neighbouring paths out of infinite no. of possibilities.

For the deductions of this principle there must be satisfied two following conditions —

δt must be zero at end points

and δr must be zero at end points.

Let the ith particle of the system be acted upon by a no. of forces given by $\vec{F_i}$ acquiring accleration $\ddot{\underline{r}}_i$, so that we have

$$\vec{F_i} = m_i \ddot{\underline{r}}_i$$

From D' Alembert's Principle, we have

$$\sum_i (\vec{F_i} - m_i \ddot{\underline{r}}_i)\delta \underline{r}_i = 0$$

$$\text{or} \quad \sum_i \vec{F_i} \cdot \delta \underline{r}_i - \sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = 0 \quad \text{—— (1)}$$

$$\text{But} \quad \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \frac{d}{dt}(\delta \underline{r}_i) \quad \text{—— (2)}$$

If there is a little variation along the actual and neighbouring paths, we have

$$\frac{d}{dt}(\delta \underline{r}_i) = \delta \frac{d}{dt}(\underline{r}_i) = \delta \dot{\underline{r}}_i \quad \text{—— (3)}$$

using eqn (3), eqn (2) may be written as

$$\ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \quad \text{—— (4)}$$

using eqn (4), eqn (1) becomes

$$\sum_i \underline{F}_i \cdot \delta \underline{r}_i - \sum_i m_i \left[\frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \right] = 0$$

or $$\sum_i \underline{F}_i \cdot \delta \underline{r}_i - \sum_i m_i \left[\frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \frac{1}{2}\delta(\dot{\underline{r}}_i^2) \right] = 0$$

or $$\sum_i \underline{F}_i \cdot \delta \underline{r}_i + \sum_i \frac{1}{2} m_i \delta(\dot{\underline{r}}_i^2) = \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)$$

or $$\sum_i \underline{F}_i \cdot \delta \underline{r}_i + \delta \sum_i \left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right) = \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i) \quad \text{—} (5)$$

But $\sum_i \underline{F}_i \cdot \delta \underline{r}_i = \delta W$ = work done by the forces $\underline{F}_i$ during displacement $\delta \underline{r}_i$

and $\delta \sum_i \left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right) = \delta T$ where T is K.E

Therefore eqn. (5) becomes

$$\delta W + \delta T = \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)$$

Integrating above eqn between limits t_1 & t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)\, dt$$

$$= \sum_i \int_{t_1}^{t_2} d(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)$$

$$= \sum_i (m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)_P^Q = 0$$

Since $\delta \underline{r}_i$ at P & Q is zero

For a Conservative System, we know that

$\delta W = -\delta V$ where V is Potential Energy

$$\therefore \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} \delta(T - V)\, dt = 0$$

$$\text{or} \quad \delta \int_{t_1}^{t_2} L\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} L\, dt = \text{extremum}$$

which is Hamilton's Principle

Hamilton's Equations of motion from Hamilton's Principle (variational Principle)

We have already introduced Hamiltonian function expressed as

$$H = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t) \quad \text{--- (3)}$$

or $$L = \sum_k p_k \dot{q}_k - H \quad \text{--- (4)}$$

so $$\delta L = \delta \sum_k p_k \dot{q}_k - \delta H$$

or $$\delta L = \sum_k \delta p_k \dot{q}_k + \sum_k p_k \delta \dot{q}_k - \delta H \quad \text{--- (5)}$$

Now putting the value of δL from eqn (5) in eqn. (2) of Variational Principle, we get

$$\int_{t_1}^{t_2} \left(\sum_k \delta p_k \dot{q}_k + \sum_k p_k \delta \dot{q}_k - \delta H \right) dt = 0 \quad \text{--- (6)}$$

Since Hamiltonian is not an explicit fn of time, then

$H = H(q_k, p_k, t)$ can be written as $H = H(q_k, p_k)$ — (7)

Differentiating eqn (7) Partially, we get

$$\delta H = \sum_k \left(\frac{\partial H}{\partial q_k} \delta q_k + \frac{\partial H}{\partial p_k} \delta p_k \right) \quad \text{--- (8)}$$

Now putting the values of δH from eqn. (8) in eqn. (6), we get

$$\int_{t_1}^{t_2} \sum_k \left(\delta p_k \dot{q}_k + p_k \delta \dot{q}_k - \frac{\partial H}{\partial q_k} \delta q_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt = 0 \quad \text{--- (9)}$$

Integrating 2nd term of eqn. (9) by Parts,

$$\int_{t_1}^{t_2} p_k \delta \dot{q}_k \, dt = \int_{t_1}^{t_2} p_k \frac{d}{dt}(\delta q_k) \, dt$$

$$= \left| p_k \delta q_k \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}(p_k)\, \delta q_k \, dt$$

$$= 0 - \int_{t_1}^{t_2} \frac{d}{dt}(p_k)\, \delta q_k \, dt$$

$$= -\int_{t_1}^{t_2} \frac{d}{dt}(p_k)\, \delta q_k \, dt \quad \text{--- (10)}$$

First term is zero because at end points $\delta q_k = 0$

Now using eqn. (10) in eqn. (9), we get

$$\int_{t_1}^{t_2} \sum_k \left(\delta p_k \dot{q}_k - \frac{d}{dt}(p_k)\, \delta q_k - \frac{\partial H}{\partial q_k} \delta q_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \left\{ \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \delta q_k \right\} dt = 0$$

In order to satisfy the above equation, each integrand must vanish as δp_k and δq_k are arbitrary variations

i.e. $$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0$$

or $$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{--- (11)}$$

and $$\dot{p}_k + \frac{\partial H}{\partial q_k} = 0$$

or $$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \text{--- (12)}$$

Equations (11) and (12) are called Hamilton's equations of motion.