Principle of least Action

Another variational Principle associated with the Hamiltonian formulation is known as the Principle of least Action. In mechanics, action is a quantity defined most generally as

$$A = \int_{t_1}^{t_2} 2T\,dt = \int_{t_1}^{t_2} \sum_k P_k \dot{q}_k\,dt \quad \text{---} \quad (1)$$

Therefore the Principle of Least Action states that in a system for which H is conserved

$$\Delta \int_{t_1}^{t_2} \sum_k P_k \dot{q}_k\,dt = 0 \quad \text{-----} \quad (2)$$

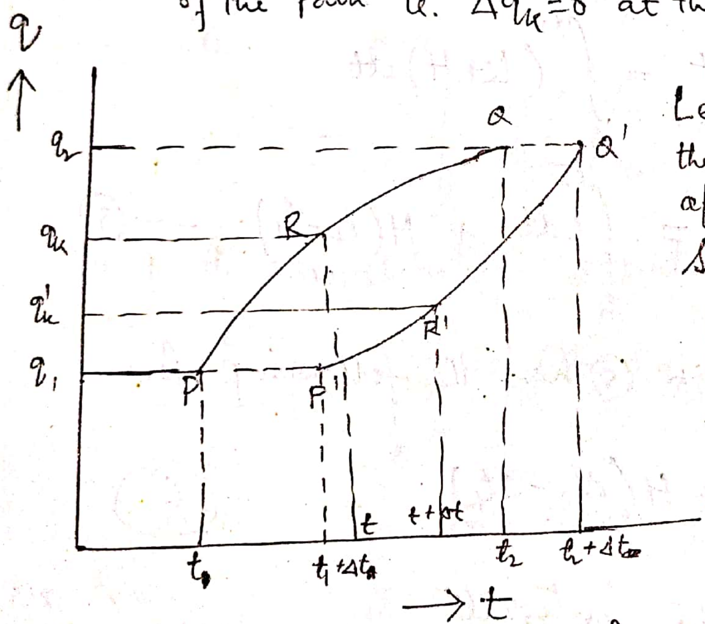
where Δ represents a new type of variation of path

In order to deduce this Principle, we have to use Δ variation in which

(I) Time as well as position Co-ordinates are allowed to vary

(II) Time varies even at end points of the Path

(III) The position Co-ordinates are held fixed at the end points of the Path i.e. $\Delta q_k = 0$ at the end points.



Let PRQ be the actual Path and P'R'Q', the varied Path. The end points P & Q after time δt takes the Position P' & Q' such that position Co-ordinates of P and Q are fixed while the time t is not fixed. A point R on actual Path now goes on R' with the Correspondence

$$q_k \to q'_k = q_k + \Delta q_k.$$

If α is the variational Parameter, then in δ process t is independent of α but in Δ process t is ~~even~~ function of α even at end points i.e. $t = t(\alpha)$

Thus the function q_k depends upon t and α throughout i.e. $q_k = q_k(t, \alpha)$

Analytically Δ variation is defined as

$$\Delta q_k = \left[\frac{d}{d\alpha} q_k(\alpha, t)\right] d\alpha = \left[\frac{\partial q_k}{\partial \alpha} + \frac{dq_k}{dt}\frac{dt}{d\alpha}\right] d\alpha$$

$$= \frac{\partial q_k}{\partial \alpha} d\alpha + \dot{q}_k \frac{dt}{d\alpha} d\alpha$$

But $\delta q_k = \frac{\partial q_k}{\partial \alpha} d\alpha$ and $\dot{q}_k \frac{dt}{d\alpha} = \dot{q}_k \Delta t$

So $\Delta q_k = \delta q_k + \dot{q}_k \Delta t$ ——— (3)

The relation beetween Δ-variation and δ-variation can now shown to hold for any function $f(q_k, t)$ as

$$\Delta f = \delta f + \dot{f} \Delta t \quad \left[\Delta f = \sum_k \frac{\partial f}{\partial q_k} \Delta q_k + \frac{\partial f}{\partial t} \Delta t \right.$$

Thus. $\Delta = \delta + \Delta t \frac{d}{dt}$ ——(4) $\quad = \sum_k \frac{\partial f}{\partial q_k} (\delta q_k + \dot{q}_k \Delta t) + \frac{\partial f}{\partial t} \Delta t$

It is to be noted that the $\quad \left. = \delta f + \dot{f} \Delta t \right]$

Δ operation and time differentiation can't be interchanged in this case which is done in δ-variation

Now from eqn. (1), we have

$$A = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k \, dt = \int_{t_1}^{t_2} (L + H) \, dt$$

$$= \int_{t_1}^{t_2} L dt + H(t_2 - t_1) \quad \text{——— (5)}$$

The Δ-variation of eqn. (5) has the following form

$$\Delta A = \Delta \int_{t_1}^{t_2} L dt + H(\Delta t_2 - \Delta t_1) \quad \text{——— (6)}$$

Let us now solve the integral

$$\Delta \int_{t_1}^{t_2} L dt$$

It is also rembembered that t_1 & t_2 limits are also subjected to change in this variation, Δ can't be taken inside the integral.

let $\int_{t_1}^{t_2} L dt = I$

So

$$\Delta I = \delta I + \dot{I}\Delta t \quad \text{from eq}^n \ 4$$

Therefore $\Delta\int_{t_1}^{t_2} L\,dt = \Delta I(t_2) - \Delta I(t_1)$

$$= \left[\delta I(t_2) + \dot{I}(t_2)\Delta t_2\right] - \left[\delta I(t_1) + \dot{I}(t_1)\Delta t_1\right]$$

$$= \delta I(t_2) - \delta I(t_1) + \dot{I}(t_2)\Delta t_2 - \dot{I}(t_1)\Delta t_1$$

$$= \delta\int_{t_1}^{t_2} L\,dt + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{——— } 7$$

From the nature of δ-variation, we have

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\left(\sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum \frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k\right)dt$$

which by Lagrange's equations can be written

$$\delta\int_{t_1}^{t_2} L\,dt = \sum_k \int_{t_1}^{t_2}\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\delta q_k)\right\}dt$$

$$= \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right)dt$$

~~8~~ with substitution equation 7 becomes

$$\Delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right)dt + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{——— } 8$$

using eqn. 3 to the first interal term of eqn 8

we have

$$\Delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_k \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\Delta t\right)dt \quad \left[\delta q_k = \Delta q_k - \dot{q}_k\Delta t\right]\right. + \left[L\Delta t\right]_{t_1}^{t_2}$$

$$= \left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\Delta q_k - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\Delta t\right]_{t_1}^{t_2} + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{——— } 9$$

At the end points $\Delta q_k = 0$ but Δt does not. So equation 9 becomes

$$\Delta\int_{t_1}^{t_2} L\,dt = \left[-\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k \Delta t\right]_{t_1}^{t_2} + \left[L\Delta t\right]_{t_1}^{t_2}$$

$$= \left[-\sum_k p_k \dot{q}_k \Delta t\right]_{t_1}^{t_2} + \left[L\Delta t\right]_{t_1}^{t_2}$$

$$= \left[\left(L - \sum_k p_k \dot{q}_k\right)\Delta t\right]_{t_1}^{t_2}$$

$$= \left[-H\Delta t\right]_{t_1}^{t_2} \qquad (10)$$

If we restrict to the system for which H is const. then $\frac{\partial H}{\partial t} = 0$

Thus $\left[H\Delta t\right]_{t_1}^{t_2} = \Delta\int_{t_1}^{t_2} H\,dt$

Substituting this in eq. (10), we get $\Delta\int_{t_1}^{t_2} L\,dt = -\Delta\int_{t_1}^{t_2} H\,dt$ *

Combining these terms, the total variation of action is

$$\Delta A = \left[\left(-\sum_k p_k \dot{q}_k + L + H\right)\Delta t\right]_{t_1}^{t_2}$$

$$= \left[(-H + H)\Delta t\right]_{t_1}^{t_2} = 0 \qquad (11)$$

This completes the proof of Principle of least action

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* $\Delta\int_{t_1}^{t_2} (L+H)\,dt = 0$

or $\Delta\int_{t_1}^{t_2} \left(\cancel{L} + \sum_k p_k \dot{q}_k - \cancel{L}\right)dt = 0$

$= \Delta\int_{t_1}^{t_2} \sum p_k \dot{q}_k\,dt = 0 \Rightarrow \Delta\int_{t_1}^{t_2} 2T\,dt = 0$

which is the Principle of least action.