Milne's Method

This method uses Newton's forward difference formula in the form

$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{\lfloor 2} \Delta^2 f_0 + \frac{n(n-1)(n-2)}{\lfloor 3} \Delta^3 f_0 \dots$ —— (1)

Substituting Eqn (1) in the relation

$y_4 = y_0 + \int_{x_0}^{x_4} f(x,y)$ —— (2)

$y_4 = y_0 + \int_{x_0}^{x_4} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2} \Delta^2 f_0 + \frac{n(n-1)(n-2)}{6} \Delta^3 f_0 + \dots \right] dx$

$= y_0 + h \int_0^4 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2} \Delta^2 f_0 + \dots \right] dn$

$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3} \Delta^2 f_0 + \frac{8}{3} \Delta^3 f_0 \right]$

$y_4 = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3)$ —— (3)

after neglecting fourth-and higher order differences and expressing differences Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ interms of the function values. we have

[P- 76, S.S Sastry]

$\Delta f_0 = f_1 - f_0$

$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = = f_2 - f_1 - f_1 + f_0$

$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = f_3 - 2f_2 + f_1 - (f_2 - 2f_1 + f_0)$

$\Delta^3 f_0 \qquad = f_3 - 3f_2 + 3f_1 - f_0$

Similarly $\Delta^4 f_0 = \Delta^3 f_1 - \Delta^3 f_0 = f_4 - 4f_3 + 6f_2 - 4f_1 + f_0$

Hence the formula in Eqn (3) can be used to predict the value of y_4 when those of y_0, y_1, y_2 and y_3 known.

To obtain a corrector formula, we substitute Newton's forward differences formula in Eq (1)

$$y_2 = y_0 + \int_{x_0}^{x_2} f(x, y)\, dx \longleftarrow (4)$$

$$y_2 = y_0 + h \int_0^2 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2} \Delta^2 f_0 + \cdots \right] dn$$

$$y_2 = y_0 + h \left[2f_0 + 2\Delta f_0 + \frac{1}{3} \Delta^2 f_0 \cdots \right]$$

$$y_2 = y_0 + \frac{h}{3} \left[f_0 + 4f_1 + f_2 \right] \quad \text{---} \quad (5)$$

The general form of Eqn (4) and (5)

$$y^p_{n+1} = y_{n-3} + \frac{4h}{3} \left[2f_{n-2} - f_{n-1} + 2f_n \right] \quad \text{---} \quad (6)$$

$$y^c_{n+1} = y_{n-1} + \frac{h}{3} \left[f_{n-1} + 4f_n + f_{n+1} \right] \quad \text{---} \quad (7)$$

Exp. We consider again the differential equation in previous example of predictor corrector formula from $\frac{dy}{dx} = 1 + y^2$ with $y(0) = 0$ and we wish compute $y(0.8)$ and $y(1.0)$ with $h = 0.2$.

Solution. The values of $y(0.2)$, $y(0.4)$ and $y(1.6)$ are computed in previous example and these are given in table

	x	y		$\frac{dy}{dx} = 1+y^2$
0	0.0	0.0	f_0	$1+(0)^2 = 1.0$
1	0.2	0.2027	f_1	$1+(.2027)^2 = 1.0411$
2	0.4	0.4228	f_2	$1+(.4228)^2 = 1.1787$
3	0.6	0.6841	f_3	$1+(.6841)^2 = 1.4681$
4	0.8	1.0239	f_4	$1+(1.0239)^2 = 2.0480$

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To obtain $y(.8)$ from Eqn (6)

$$y_4^p = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$y_4^p = y(.8) = 0 + \frac{4\times .2}{3}\left[2\times(1.0411) - 1.1787 + 2\times(1.4681\right]$$

$$y^p(0.8) = 1.0239$$

This gives
$$\frac{dy}{dx} = 1 + y^2$$

$$y'(0.8) = 1 + (1.0239)^2 = 2.0480$$

To correct this value of $y(0.8)$ by using formula from Eqn (7)

$$y^c(0.8) = 0.4228 + \frac{0.2}{3}\left[1.1787 + 4(1.4681) + 2.0480\right]$$

$$y(0.8) = 1.0294$$

Proceeding similarly we obtain $y(1.0)$ using Eqn (6) and (7)

$$y_{3+1} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4\times .2}{3}\left[2\times 1.787 - 1.4681 + 2\times 2.0480\right]$$

$$y^p(1.0) = 1.3297 \qquad y'(1.0) = 1 + (1.3397)^2$$

To obtain correct value $\qquad = 2.7841$

$$y^c(1.0) = .6841 + \frac{0.2}{3}\left[1.4681 + 4\times(2.0597) + 2.7841\right]$$

$$y(1.0) = 1.5549$$