LINEAR PROGRAMMING

Some Definitions and Notations

Matrix form of General LP Problem

Let the L.P. problem after introducing the slack or surplus variables be as follows:

$$\text{Max. } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0\cdot x_{n+1} + 0\cdot x_{n+2} + \cdots + 0\cdot x_{n+m}$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n + x_{n+1} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + a_{2n}x_n + x_{n+2} = b_2$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n + x_{n+m} = b_m$$

$x_i \geq 0$ for all $i = 1, 2, 3, \ldots n, n+1, \ldots, n+m$;
$b_1, b_2, \ldots, b_m$ are all positive.

It has to be noted that the coefficients of slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ in the objective function are assumed to be zero so that the conversion into a system of linear equation does not change the function to be optimised.

The above L.P. problem may also be stated as

$$\text{Max. } Z = cx$$

Subject to $Ax = b$

and $x \geq 0$

where $c = (c_1, c_2, \ldots, c_n, 0, 0, 0, \ldots 0)_{1\times(m+n)}$ a row vector of order $1\times(m+n)$.

$A = [a_{ij}]_{m \times (m+n)}$ the matrix of Coefficients of order $m \times (m+n)$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}_{(n+1) \times 1} = [x_1, x_2, \dots x_n, x_{n+1}, \dots, x_{n+m}]$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1} = [b_1, b_2, \dots b_m]$$

If α_j denotes the jth Column of the matrix A then

$$A = (\alpha_1, \alpha_2, \dots, \alpha_{n+m}).$$

(i) BASIS MATRIX :-

First we shall denote by B a $m \times n$ non-singular matrix whose Column vectors are m number of linearly independent Columns of the matrix A. If these Columns are denoted by $\beta_1, \beta_2, \dots, \beta_m$ then

$$B = (\beta_1, \beta_2, \dots, \beta_m)$$

Matrix B is called the basis matrix.

(ii) The variables corresponding to $\beta_1, \beta_2, \dots, \beta_m$ Called the basic variables are denoted by $x_{B_1}, x_{B_2}, x_{B_m}$ respectively.

The vector of these m basic variables is denoted by x_B i.e.

$$x_B = [x_{B_1}, x_{B_2}, \dots, x_{B_m}]$$

where $x_B = B^{-1}b$

and is called the B.F.S. of the L.P.P.

(iii) The coefficients of basic variables $x_{B_1}, x_{B_2}, \dots, x_{B_m}$ in the objective function Z will be denoted by $c_{B_1}, c_{B_2}, \dots, c_{B_m}$, so that

$$C_B = (c_{B_1}, c_{B_2}, c_{B_m})$$

(iv) Since $B = (\beta_1, \beta_2, \dots, \beta_m)$ is a non-singular matrix of order $m \times m$, and the vector $\beta_1, \beta_2, \dots, \beta_m$ are linearly independent, they form the basis of E^m. Therefore each vector in E^m can be expressed as a linear Combination of vectors in B.

Let $\alpha_j = \beta_1 y_{1j} + \beta_2 y_{2j} + \dots + \beta_m y_{mj}$

$$= (\beta_1, \beta_2, \dots, \beta_m) \cdot \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{bmatrix}$$

$$= BY_j, \quad Y_j = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{bmatrix}_{m\times 1}$$

$y_{1j}, y_{2j}, \dots, y_{mj}$ are scalars required to express α_j, $(j = 1, 2, 3, \dots, n+m)$ as Linear Combination of $\beta_1, \beta_2, \dots, \beta_m$.

$$\therefore Y_j = B^{-1}\alpha_j.$$

(v) In the end we write

$$Z_j = C_B Y_j = c_{B1} y_{1j} + c_{B2} y_{2j} + \dots + c_{Bm} y_{mj}$$

Definition :- The form max $Z = CX$

Subject to the constraints

$$AX = b$$
$$X \geq 0$$

is known as the standard form of L.P. problem.

Thus we have the following characteristics of the standard form of LP problem

(i) The objective function of maximization type

(ii) All constraints are expressed as equations.

(iii) R.H.S. of each constraint is non-negative.

(iv) All variables are non-negative.

In a general linear programming problem, it is assumed that the number of rows of the coefficient matrix A is less than the number of columns.

Notes :- Similar treatment can be adopted in the case of surplus variables.