Linear Combination - Let $V(F)$ be a vector space over Field F. If $v_1, v_2, \ldots v_n \in V$ then any vector

$$V = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$$

where $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n \in F$

is called a Linear combination of vectors $v_1, v_2 \ldots v_n$.

Linear Span Let $V(F)$ be a vector space and S be any non-empty subset of V. Then the linear span of S is the set of all Linear combination of finite sets of elements of S and is denoted by $L(S)$.

$$L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n : \alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$$
is any arbitrary finite subset of F
and $v_1, v_2, \ldots v_n$ is finite subset of $S\}$

Exp. The subset $\{(1,0,0), (0,1,0)\}$ of $V_3(F)$ generates the subspace which is the totality of the elements of the form $(a,b,0)$

$$L(S) = a(1,0,0) + b(0,1,0)$$
$$= (a,0,0) + (0,b,0)$$
$$L(S) = (a,b,0)$$

Exp- The subset $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $V_3(F)$ generates or spans the entire vector space $V_3(F)$ i.e $L(S) = V$

If (α, β, γ) be any element of F, then

$$L(S) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$$
$$= (\alpha,0,0) + (0,\beta,0) + (0,0,\gamma)$$
$$L(S) = (\alpha, \beta, \gamma)$$

Thus $(\alpha, \beta, \gamma) \in L(S)$

$V \subseteq L(S)$ and Also $L(S) \subseteq V$

Hence $L(S) = V$

Linear Dependence & Independence of Vectors.-

L.D.→Definition- Let $V(F)$ be a vector space over a field F. Then finite set $\{v_1, v_2, v_3 \cdots v_n\}$ of vectors of V is said to be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n$ not all of them equal to zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n = 0$$

L.I.→Definition- Let $V(F)$ be a vector space over F. Then a finite set of vectors $\{v_1, v_2 \cdots v_n\}$ of V is said to linearly independent if for every expressions of the type-

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n = 0$$

where $\alpha_1 = 0 = \alpha_2 = \alpha_3 = \cdots = \alpha_n$. $\quad \alpha_i \in F$ & $v_i \in V$

* Any set of vectors in V is said to be linearly dependent (or linearly independent), if every finite subset of S is linearly dependent (or linearly independent).

Remark- ① If $v \neq 0 \in V(F)$, then the singleton $\{v\}$ is linearly independent, since for any $\alpha \in F$, $\alpha v = 0 \Rightarrow \alpha = 0$

② The set $\{0\}$ is linearly dependent, since for each $\alpha \neq 0 \in F$, $\alpha 0 = 0$

(3) Any set of vectors containing zero vector is linearly dependent since for the set $\{v_1, v_2, \cdots v_n, 0\}$. $0.v_1 + 0v_2 + 0v_3 + \cdots 0v_n + \alpha 0 = 0$ $(\alpha \neq 0 \in F)$.

⇒ <u>Linearly dependent vectors</u> → if the determinant of vectors i.e $|A| = 0$ then vectors are dependent i.e $\alpha_1 = \alpha_2 = \alpha_3 = \dots \neq 0$ (not all zero)

or

(*) rank of vectors matrix < no. of vectors.

⇒ Linearly independent if $|A| \neq 0$ i.e $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (all zero)

(*) rank of matrix of vectors = no. of vectors.

Vectors written in matrix form
$V_1 = (1, 2, 1)$, $V_2 = (3, 1, 5)$ & $V_3 = (3, -4, 7)$

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} & \\ \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix} & \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \end{array}$$

(zero) make

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 2 & 4 \end{bmatrix} \quad R_3 \to 5 \times R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \qquad |A| = 1(0) - 3(0) + 3(0)$$

$\cancel{= 15 \neq 0}$

rank of $A = 2 \Rightarrow \rho(A)$