Linear differential Equation–

A differential equation of the form $\frac{dy}{dx} + Py = Q$ — (i)

where, P and Q are constants or functions of x only and first order differential equation.

or form may be

$$\frac{dx}{dy} + P_1 x = Q_1 \quad \text{— (ii)}$$

where P_1 and Q_1 are constants or functions of y only.

Exp. $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^{x}$ → This L.D.E in form (i)

or $\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$ → This L.D.E in form (ii)

To solve the first order linear differential equation of the type $\frac{dy}{dx} + Py = Q$ — (1)

multiply both sides of the equation by $g(x)$ to get.

$$g(x)\frac{dy}{dx} + P(g(x))y = Q.g(x) \quad \text{— (2)}$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y.g(x)$.

i.e $g(x)\frac{dy}{dx} + P.g(x)\,y = \frac{d}{dx}[y.g(x)]$

$$g(x)\frac{dy}{dx} + P.g(x)\,y = g(x)\frac{dy}{dx} + y\,g'(x)$$

$$\Rightarrow P(g(x)) = g'(x)$$

$$P = \frac{g'(x)}{g(x)}$$

Integrating both sides w.r.to x, we get

$$\int P\,dx = \int \frac{g'(x)}{g(x)}\,dx$$

$$\int P\,dx = \log[g(x)]$$

$$\Rightarrow g(x) = e^{\int P\,dx}$$

On multiplying the Eqn (1) by $g(x) = e^{\int P\,dx}$, the L.H.S becomes the derivative of some function of x and y. The function $g(x) = e^{\int P\,dx}$ is called Integrating factor (I.F) of the given diff. Eqn.

Substituting the value of $g(x)$ in equation (2)

$$e^{\int P\,dx}\frac{dy}{dx} + P e^{\int P\,dx} y = Q \cdot e^{\int P\,dx}$$

$$\frac{d}{dx}\left(y \cdot e^{\int P\,dx}\right) = Q \cdot e^{\int P\,dx}$$

Integrating both sides w.r.to x

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx + c$$

which is the general solution of the differential equation.

⇒ Steps for Solving first order Linear differential equation.

(i) Write diff. Eqn. in the form $\frac{dy}{dx} + P(x)y = Q(x)$ and compare $P(x)$ and $Q(x)$.

(ii) Find the I.F $= e^{\int P(x)dx}$

(iii) Write the solution of the given differential equation as

$$y(I.F) = \int (Q(x) \cdot I.F)\, dx + C,$$

and solve it.

Exp. Find the particular solution of diff. Eqn.

$$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x \quad (x \neq 0)$$

given that $y=0$ when $x = \pi/2$

Sol. Given diff. Eqn.

$$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x$$

of the type $\frac{dy}{dx} + P(x)y = Q(x)$

$P(x) = \cot x$

$Q(x) = 2x + x^2\cot x$

Therefore

$I.F = e^{\int P(x)dx}$

$I.F = e^{\int \cot x\, dx} = e^{\log \sin x}$

$I.F = \sin x$

Hence, the solution of differential Eqn.

$$y \cdot \sin x = \int (2x + x^2 \cot x) \cdot \sin x \, dx + c$$

$$= \int (2\underset{II}{x} \underset{I}{\sin x}) \, dx + \int x^2 \cos x \, dx + c$$

$$= \sin x \left(\frac{2x^2}{2}\right) - \int \cos x \cdot \left(\frac{\cancel{2}x^2}{\cancel{2}}\right) dx$$

$$+ \int x^2 \cos x \, dx + c$$

$$y \sin x = x^2 \sin x + c \quad \text{——} \quad (1)$$

put $y = 0$ and $x = \pi/2$ in Eq (1)

$$0 = \frac{\pi^2}{4} \sin \pi/2 + C$$

$$c = -\frac{\pi^2}{4}$$

Then

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

$$y = x^2 - \frac{\pi^2}{4 \sin x} \quad (\because \sin x \neq 0)$$