Limit form Test:-

If two positive term series Σa_n and Σb_n be such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \text{, a finite quantity } (\neq 0)$$

then Σa_n and Σb_n converges or diverges together.

Proof:-

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, a finite number $(\neq 0)$

By definition of a limit, there exists a positive number ϵ, however small, such that

$$\left| \frac{a_n}{b_n} - l \right| < \epsilon \quad \text{for } n \geq m$$

or $$-\epsilon < \frac{a_n}{b_n} - l < \epsilon \quad \text{for } n \geq m$$

or $$l - \epsilon < l + \frac{a_n}{b_n} - l < \epsilon + l \quad \forall\, n \geq m$$

$$l - \epsilon < \frac{a_n}{b_n} < \epsilon + l \quad \forall\, n \geq m$$

Omitting the first m terms of both series, we have

$$l - \epsilon < \frac{a_n}{b_n} < l + \epsilon \quad \text{for all } n \text{ where } b_n > 0$$

$$(l-\epsilon)\, b_n < a_n < (l+\epsilon)\, b_n \text{, for all } n \quad \text{——(1)}$$

Case (i) - When $\sum b_n$ is convergent then

$$\lim_{n\to\infty} (b_1 + b_2 + \cdots\cdots + b_n) = k, \text{ a finite n.}$$ ——— (2)

from (1)

$$\frac{a_n}{b_n} < l + \epsilon \quad \text{i.e.} \quad a_n < (l + \epsilon) b_n \text{ for all}$$

$$\therefore \lim_{n\to\infty} (a_1 + a_2 + \cdots\cdots + a_n) < (l + \epsilon) \lim_{n\to\infty} (b_1 + b_2 + \cdots$$

$$< (l + \epsilon) k \quad \text{from (2)}$$

Hence $\sum a_n$ is also convergent.

Case (ii) When $\sum b_n$ is divergent then

$$\lim_{n\to\infty} (b_1 + b_2 + \cdots\cdots + b_n) \to \infty. \quad \text{——— (3)}$$

from (1)

$$(l - \epsilon) < \frac{a_n}{b_n} \quad \text{or} \quad a_n > (l - \epsilon) b_n \ \forall n$$

$$\lim_{n\to\infty} (a_1 + a_2 + \cdots\cdots + a_n) > (l - \epsilon) \lim_{n\to\infty} (b_1 + b_2 + \cdots$$

$$\lim_{n\to\infty} (a_1 + a_2 + \cdots\cdots + a_n) > (l - \epsilon) \infty \to \infty$$

from (3)

Hence $\sum a_n$ is also divergent.

Exp. Test for convergence series.

$$\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\cdots\cdots$$

Solution Given series.

$$\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\cdots\cdots$$

The n^{th} term is a_n

$$a_n = \frac{1}{\sqrt{n(n+1)}} = \frac{1}{n\sqrt{(1+1/n)}}$$

let $b_n = \frac{1}{n}$

then $\frac{a_n}{b_n} = \frac{1}{\sqrt{(1+1/n)}}$

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+1/n}}\right) = \frac{1}{\sqrt{1+0}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ and finite & non-zero

$\because \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

So both $\sum a_n$ & $\sum b_n$ converge or diverge together.

Since $\sum b_n = \sum \frac{1}{n}$ is known to divergent becau

$\left(\because \lim_{n\to 0}\left(\frac{1}{n}\right) = \infty\right)$, So $\sum a_n$ is also divergent