Cosets and Lagrange's theorem

Cosets :- Let H be a subgroup of a group G. For any $a \in G$, the set $aH = \{ah : h \in H\}$ is called a left coset of H in G. Similarly the set $Ha = \{ha : h \in H\}$ is called a right coset of H in G.

Since $eH = He = H$, it is clear that H is itself a left coset as well as a right coset of H in G. Again since $e \in H$, we have $a \in aH$ and $a \in Ha$.

Remark :- (i) If H is a subgroup of an abelian group G and $a \in G$ then $aH = Ha$ and thus for an abelian group, every left coset is a right coset and conversely.

(ii) In case $aH = Ha$ for all elements a for a group G (where H is a subgroup of G) we will refer to aH simply as a coset of H in G.

(iii) Let H be a subgroup of G and let aH be the left coset of H in G. Then if $b \in aH$, $bH = aH$. Thus every element in aH is a representative of aH.

(iv) Let H be a subgroup of a group G, and let aH and bH be any two left cosets of H in G. Then either $aH = bH$ or $aH \cap bH = \phi$.

(v) Let H be a subgroup of group G. Then any two left cosets of H in G have the same cardinality.

Lagrange's theorem :-

If G is a finite group of order n, the order of every subgroup H of G is a divisor of n.

Proof:- Since the group G has n elements there are only a finite number, say $k (\leq n)$ of distinct left cosets of H in G. Let distinct left cosets of H in G be denoted by

$a_1 H = eH = H,\ a_2 H, \ldots\ a_k H$

$G = a_1 H \cup \cdots \cup a_k H \ldots$ (α)

$a_i H$ are mutually disjoint $(i = 1, 2, \ldots k)$.

Any two $a_i H$ have the same number of elements. Hence each $a_i H$ has the same number of elements as $H = a_1 H$. Then it follows from (α) that

$$O(G) = K \cdot O(H)$$

Hence $O(H)$ = divs $O(G) = n$.

Remark :- It is clear from the above proof that the number of distinct left cosets of H in G is equal to $O(G)/O(H)$.

Now, it is evident that $\frac{o(G)}{o(H)}$ is equal to the number of right cosets of H in G. Similarly it can be proved that $\frac{o(G)}{o(H)}$ is equal to the number of left cosets of H in G.