* Iteration Method →

We already describe methods which require one or more approximate values to start the solution and these values need not necessarily bracket the root. The first is the iteration method which requires one starting value of x.

Now, we describle this method for finding a root of the equation

$$f(x)=0 \quad \text{—— (1)}$$

We rewrite this equation in the form

$$x=\phi(x) \quad \text{—— (2)}$$

for example, the equation $x^3+x^2-2=0$ can be expressed in different forms (ways)

$$x=\sqrt{\frac{2}{1+x}}, \quad x=\sqrt{2-x^3}, \quad x=(2-x^2)^{1/3}, \text{ etc.}$$

Now, let x_0 be an approximate root of Eqn (2). Then, substituting in Eqn (2), we get the first approximation

$$x_1=\phi(x_0)$$

Successive substitutions give the approximations $x_2=\phi(x_1)$, $x_3=\phi(x_2)$ ----- , $x_n=\phi(x_{n-1})$

The preceding sequence may not converge to a definite number. But if the sequence converges to a definite number $\xi(xi)$, then ξ will be a root of the equation $x=\phi(x)$. To show this

Let $x_{n+1}=\phi(x_n)$ —— (3)

be the relation between nth and $(n+1)$th approxs.

As n increases, $x_{n+1} \to \xi$ and if $\phi(x)$ is a continuous function, then $\phi(x_n) \to \phi(\xi)$.
Hence, in the limit, we obtain

$$\xi = \phi(\xi) \quad \text{——} \quad (4)$$

Which shows that ξ is a root of the equation $x = \phi(x)$.
We proceed in the following way:
We have from Eqn

$$x_1 = \phi(x_0) \quad \text{——} \quad (5)$$

From Eqn (4) & (5) we get

$$\xi - x_1 = \phi(\xi) - \phi(x_0)$$

$$\xi - x_1 = (\xi - x_0)\phi'(\xi_0), \quad x_0 < \xi_0 < \xi \quad \text{——} \quad (6)$$

Similarly we obtain

$$\xi - x_2 = (\xi - x_1)\phi'(\xi_1), \quad x_1 < \xi_1 < \xi \quad \text{——} \quad (7)$$

$$(\xi - x_3) = (\xi - x_2)\phi'(\xi_2), \quad x_2 < \xi_2 < \xi \quad \text{——} \quad (8)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\xi - x_{n+1} = (\xi - x_n)\phi(\xi_n), \quad x_n < \xi_n < \xi \quad \text{——} \quad (9)$$

We assume $|\phi'(\xi_i)| \le k$ (for all i)

then Eqns 6 to 9 give

$$\left.\begin{aligned} |\xi - x_1| &\le k|\xi - x_0| \\ |\xi - x_2| &\le k|\xi - x_1| \\ |\xi - x_3| &\le k|\xi - x_2| \\ &\vdots \\ |\xi - x_{n+1}| &\le k|\xi - x_n| \end{aligned}\right\} \quad \text{——} \quad (10)$$

Multiplying the corresponding sides of the above/ previous Eqns., we obtain

$$|\xi - x_{n+1}| \leq k^{n+1} |\xi - x_0| \quad \text{——} \quad (11)$$

if $k \leq 1$, i.e if $|\phi'(\xi_i)| < 1$, then the right side of equation (11) tends to zero and the sequence of approximations $x_0, x_1, x_2 \cdots \cdots$ converges to the root ξ. Thus when we express the equation $f(x) = 0$ in the form $x = \phi(x)$, then $\phi(x)$ must be such that

$$|\phi'(x)| < 1$$

in an immediate neighbourhood of the root. It follows that if the initial approximation x_0 is chosen in an iterval containing the root ξ, then the sequence of approximations converges to the root ξ.

Now we show that the root so obtained is unique. To prove this, let ξ_1 and ξ_2 be two roots of the equation $x = \phi(x)$. Then we must have

$$\xi_1 = \phi(\xi_1) \;\&\; \xi_2 = \phi(\xi_2)$$

Therefore

$$|\xi_1 - \xi_2| = \phi(\xi_1) - \phi(\xi_2)$$
$$= |\xi_1 - \xi_2| \phi'(\eta), \; \eta \in (\xi_1, \xi_2)$$

Hence

$$|\xi_1 - \xi_2| [1 - \phi'(\eta)] = 0 \quad \text{——} \quad (12)$$

Since $|\phi'(\eta)| < 1$, it follows that $\xi_1 = \xi_2$, which proves that the root obtained is unique.

Finally, we shall find the error in the root obtained.
We have

$$|\xi - x_n| \le |\xi - x_{n-1}|$$

$$= k|\xi - x_n + x_n - x_{n-1}|$$

$$\le k[|\xi - x_n| + |x_n - x_{n-1}|]$$

$$\Rightarrow |\xi - x_n| \le \frac{k}{1-k}|x_n - x_{n-1}|$$

$$\le \frac{k}{1-k} k^{n-1}|x_1 - x_0|$$

$$|\xi - x_n| \le \frac{k^n}{1-k}|x_1 - x_0| \quad \text{—— (13)}$$

Which shows that all the convergence would be faster for smaller values of k.
Now, let ε be the specified accuracy so that

$$|\xi - x_n| \le \epsilon$$

Then Eqn (13) gives

$$|x_n - x_{n-1}| \le \frac{1-k}{k}\epsilon \quad \text{—— (14)}$$

Which can be used to find the difference between two successive approximations (or iterations) to achieve a prescribed accuracy.