GROUP THEORY

INTERNAL DIRECT PRODUCTS :-

A group G is said to be the internal direct product of two of its subgroups H_1 and H_2 if

(i) every element of H_1 commutes with every elements of H_2

(ii) every element of G is uniquely expressible as the product of an element of H_1 and an element of H_2.

Also in this case we write $G = H_1 \times H_2$.

THEOREMS ON INTERNAL DIRECT PRODUCT OF GROUPS :-

Theorem :- Let a group G be the internal direct product of two of its subgroups H_1 and H_2. Then

(i) The identity element e of G is the only element common to both H_1 and H_2

(ii) $G = H_1 \times H_2$

Proof :- Let $a \in G$ be an element common to both H_1 and H_2.

Since H_1 and H_2 are subgroups of G so a^{-1} will belong to both H_1 and H_2.

G is the internal direct product of H_1 and H_2, then an arbitrary element $x \in G$ can be uniquely expressed as the product of an element of H_1 and an element of H_2.

So, $x = h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$.

Also we may write

$$x = (h_1 a)(a^{-1} h_2) \text{ where } h_1 a \in H_1 \text{ and}$$
$a^{-1} h_2 \in H_2$.

But the expression of x being unique, we have

$$h_1 a = h_1 \Rightarrow a = e.$$

Hence the identity $e \in G$ is the only element common to both H_1 and H_2.

(ii) Since G is the internal direct product of H_1 and H_2, then $x \in G$ is uniquely expressible as the product of an element of H_1 and an element of H_2.

i.e. $x = h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$.

Now define a mapping $\phi : G \to H_1 \times H_2$ by

$$\phi(x) = (h_1, h_2) \quad \forall \; x = h_1 h_2 \in G$$

ϕ is one-one. Let x and y be any two elements of G such that $x = h_1 h_2$ and $y = g_1 g_2$ for some $h_1, g_1 \in H_1$ and $h_2 g_2 \in H_2$.

Assuming that $\phi(x) = \phi(y)$

$$\Rightarrow (h_1 \cdot h_2) = (g_1 \cdot g_2) \Rightarrow h_1 = g_1 \text{ and } h_2 = g_2$$

$$\Rightarrow h_1 h_2 = g_1 g_2 \Rightarrow x = y$$

Thus ϕ is one-one.

ϕ is onto. Let (h_1, h_2) be an arbitrary element of $H_1 \times H_2$, so $h_1 \in H_1$ and $h_2 \in H_2$, then $h_1 h_2$ is an element of G such that

$$\phi(h_1 h_2) = (h_1, h_2).$$

Thus ϕ is onto.

ϕ preserves the Composition :- Let x and y be any two element of G such that $x = h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$ and $y = g_1 g_2$ for some $g_1 \in H_1$ and $g_2 \in H_2$

Now $\phi(xy) = \phi(h_1 h_2 g_1 g_2)$

$= \phi(h_1 g_1 h_2 g_2)$ [$\because$ each element of H_1 Commutes with each element of H_2]

$= (h_1 g_1 \cdot h_2 g_2)$ [By the definition of ϕ]

$= (h_1 \cdot h_2)(g_1 \cdot g_2) = \phi(x)\phi(y)$

Thus ϕ preserves the Composition

Hence ϕ is an isomorphism of G onto $H_1 \times H_2$

$G_1 = H_1 \times H_2$