Homogeneous differential Equation

* A differential equation of the form

$$\frac{dy}{dx} = F(x, y) = g(y/x)$$ is said to be homogeneous if $F(x,y)$ is a homogeneous function of degree zero.

To solve a homogeneous differential equation

$$\frac{dy}{dx} = F(x, y) = g(y/x) \quad \text{— (i)}$$

To substitution $y = v \cdot x$ —— (ii)

Differentiating equation (ii) w.r.t. x

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx} \quad \text{—— (iii)}$$

Put value of $\frac{dy}{dx}$ from Eq (iii) in Eq (i)

$$v + x\frac{dv}{dx} = g(v)$$

$$x\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Integrating both sides then

$$\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + c$$

This equation gives general solution of the differential eq (i) when we replace v by y/x.

Radha!

First-order Homogeneous Differential Equation -

A differential equation: $P(x,y)\,dx + Q(x,y)\,dy = 0$ is called homogeneous, if $P(x,y)$ and $Q(x,y)$ are homogeneous functions of the same degree.

Exp. Find the general solution to the equation

$$\frac{dy}{dx} = e^{y/x} + \frac{y}{x} \quad \text{——— (1)}$$

$$F(x,y) = e^{y/x} + \frac{y}{x}$$

$$\Rightarrow F(\lambda x, \lambda y) = e^{\lambda y/\lambda x} + \frac{\lambda y}{\lambda x}$$

$$= \lambda^0 (e^{y/x} + y/x)$$

$$= \lambda^0 F(x,y)$$

So, $F(x,y) = e^{y/x} + y/x$ is a homogeneous function of degree zero. So equation (1) is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eqn (1)

then

$$v + x\frac{dv}{dx} = e^{v} + v$$

$$\frac{dv}{e^{v}} = \frac{dx}{x}$$

Integrating

$$\int e^{-v}\,dv = \int \frac{dx}{x} + c$$

$$(e^{-v})(-1) = \log|x| + \log|c|$$

$$-e^{-v} = \log(|x \cdot c|)$$

$$\Rightarrow e^{-v} + \log|xc| \Rightarrow e^{-y/x} + \log|xc|$$

or $y = -x \ln(\log|c\,x|)$.

Exp. Solve the differential Equation.

$$\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0 \quad -(1)$$

Sol. Compare eqn. (1) from $P(x,y)dx + Q(x,y)dy = 0$

$$\Rightarrow P(x,y) = x\sin\frac{y}{x} - y\cos\frac{y}{x}$$

$$Q(x,y) = x\cos\frac{y}{x}$$

Replace $x = \lambda x$ & $y = \lambda y$ in both functions.

$$P(\lambda x, \lambda y) = \lambda x\sin\frac{\lambda y}{\lambda x} - \lambda y\cos\frac{\lambda y}{\lambda x}$$

$$= \lambda\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)$$

$$= \lambda^1 [P(x,y)]$$

Similarly

$$Q(\lambda x, \lambda y) = \lambda x\cos\frac{\lambda y}{\lambda x}$$

$$= \lambda^1 Q(x,y)$$

Therefore, $P(x,y)$ and $Q(x,y)$ are homogeneous functions of degree one. Thus eqn (1) is homogeneous differential equation.

Then putting $y = xv$ in Eqn (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$(x\sin v - xv\cos v)dx + (x\cos v)(v\,dx + x\,dv) = 0$$

$$x\sin v\,dx + x^2\cos v\,dv = 0$$

$$x(\sin v\,dx + x\cos v\,dv) = 0$$

$$\sin v\,dx = -x\cos v\,dv$$

$$\frac{dx}{x} = -\frac{\cos v}{\sin v}\,dv$$

$$\frac{dx}{x} = -\cot v\,dv \quad \text{—(ii)}$$

Integrating both sides

$$\int \frac{dx}{x} + c = -\int \cot v\,dv$$

$$\log|x| + \log c = \log|\sin v|$$

$$\log|x \cdot c| = \log|\sin v|$$

$$x \cdot c = \sin v \qquad \text{put } v = y/x$$

Then $\boxed{cx = \sin y/x}$

Alternatively from Eq ①

$$\frac{dy}{dx} = \frac{y\cos y/x - x\sin y/x}{x\cos y/x}$$

$$\frac{dy}{dx} = F(x,y)$$

$$F(x,y) = \frac{y\cos y/x - x\sin y/x}{x\cos y/x}$$

Replacing x by λx & y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda(y\cos y/x - x\sin y/x)}{\lambda x\cos y/x}$$

$$F(\lambda x, \lambda y) = \lambda^0 F(x,y)$$

Thus $F(x,y)$ is a homogeneous function of degree zero, So this is homogeneous

→ differential equation. Then substituting

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus

$$v + x\frac{dv}{dx} = \frac{vx\cos v - \frac{y}{v}\sin v}{\frac{y}{v}\cos v}$$

$$v + x\frac{dv}{dx} = v - \tan v$$

$$x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \frac{dx}{x} = -\cot v\,dv$$, This is same as Eq

Integrating both sides

$$\int \frac{dx}{x} + \log c = -\int \cot v\,dv$$

$$\log x + \log c = \log|\sin v|$$

$$x \cdot c = \sin v$$

$$x \cdot c = \sin y/x$$

Exp. Solve the Equation.

$$(x^3+y^3)\,dx - xy^2\,dy = 0$$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2} \quad \text{---} (1) \Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2} + \frac{y}{x}$$

$$\frac{dy}{dx} = f(x,y)$$

$$\Rightarrow f(x,y) = \frac{x^3+y^3}{xy^2}$$

Replacing $x \to \lambda x$ & $y \to \lambda y$ then

$$f(\lambda x, \lambda y) = \frac{\lambda^3(x^3+y^3)}{\lambda^3(xy^2)} = \lambda^0 f(x,y)$$

Hence, this equation is homogeneous differential Eqn of degree zero.

Put $y = vx$ in Eqn (1) $\Rightarrow x/y = \frac{1}{v}$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v^{-2} + v$$

$$x\frac{dv}{dx} = \frac{1}{v^2}$$

$$v^{+2}\,dv = \frac{1}{x}\,dx$$

Integrating both sides

$$\int v^{+2}\,dv = \int \frac{1}{x}\,dx + c$$

$$+\frac{v^3}{3} = \log|x| + \log|c|$$

$$\Rightarrow \frac{1}{3}\frac{y^3}{x^3} = \log|x\cdot c| \Rightarrow y^3 = 3x^3 \log|x\cdot c|$$