Imp.
**Exp.** Let $H$ be a hermitian form on inner product space $V$ over field $F$ ($R$ or $C$). Then Linear mapping/operator $T: V$ into $V$ itself is $H$-unitary or unitary operator if and only if $H(T(x), T(y)) = H(x, y)$ $\forall\, x, y \in V$.

and $H(T(x), T(x)) = H(x, x)$ $x \in V$

**Proof** By the definition of a unitary operator

Let $x, y \in V$

$H(T(x), T(x)) = H(x, x) \quad \forall\, x \in V$

or $H(T(x), T(y)) = H(x, y) \quad \forall\, x \in V, \ \forall\, y \in V$

Now

$$H(T(x+y), T(x+y)) = H(T(x), T(x)) + H(T(x), T(y)) + H(T(y), T(x)) + H(T(y), T(y))$$

By unitary defition.

$$H((x+y), (x+y)) = H(x,x) + H(T(x), T(y)) + H(T(y), T(x)) + H(y, y)$$

$$\underline{H(x,x)} + H(x,y) + H(y,x) + \underline{H(y,y)} = \underline{H(x,x)} + H(T(x), T(y)) + H(T(y), T(x)) + \underline{H(y,y)}$$

Then

$$H(x,y) + H(y,x) = H(T(x), T(y)) + H(T(y), T(x)) \quad \text{—} (1)$$

(i) If the field of scalars is real then $H$ is symmetric i.e. $H(x,y) = H(y,x)$ or $H(T(x), T(y)) = H(T(y), T(x))$

Then Eqn (1) we get

$$H(x,y) + H(x,y) = H(T(x),T(y)) + H(T(x),T(y))$$

$$2H(x,y) = 2H(T(x),T(y))$$

$$\therefore H(x,y) = H(T(x),T(y))$$

Hence proved.

(ii) If the field is complex, replacing $x$ by $ix$ in Eqn ① Then we get

$$H(ix,y) + H(y,ix) = H(T(ix),T(y)) + H(T(y),T(ix))$$

$$iH(x,y) + \bar{i}H(y,x) = iH(T(x),T(y)) + \bar{i}H(T(y),T(x))$$

$$i[H(x,y) - H(y,x)] = i[H(T(x),T(y)) - H(T(y),T(x))]$$

$$H(x,y) - H(y,x) = H(T(x),T(y)) - H(T(y),T(x)) \quad \text{— ②}$$

Adding Eqn ① & ②

$$2H(x,y) = 2H(T(x),T(y))$$

$$H(x,y) = H(T(x),T(y))$$

Hence proved