* Solution of Linear Systems

The solutions of a linear system of equations can be accomplished by a numerical method which falls in one of two categories—: direct or iterative methods.

→ In Direct methods, we describe the elimination method by Gauss as also its modification and the LU decomposition method.

→ About the iterative types, we describe only the Jacobi and Gauss-Seidel methods

Gauss-Elimination

This is the elementary elimination method and it reduces the system of equations to an equivalent upper-triangular system, which can be solved by back substitution.

Let the system of n times linear equations in n unknowns be given by—

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots a_{1n}x_n = b_1 \quad (i) \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots a_{2n}x_n = b_2 \quad (ii) \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots a_{nn}x_n = b_n \quad (iii)
\end{array}\right\} \quad (1)$$

There are two steps in the solution of

the system given equation, via the elimination of unknowns and back substitution.

Step 1 - The unknowns are eliminated to obtain an upper-triangular system.

To eliminate x_1 from the second Eqn we multiply the first equation by $(-a_{21}/a_{11})$ and obtain

$$-a_{21}x_1 - a_{12}\frac{a_{21}}{a_{11}}x_2 - \cdots a_{1n}\frac{a_{21}}{a_{11}}x_n = -b_1\frac{a_{21}}{a_{11}}$$

Adding the above equation to the second Eqn of Eqn (1)

$$\left(a_{22} - a_{12}\frac{a_{21}}{a_{11}}\right)x_2 + \left(a_{23} - a_{13}\frac{a_{21}}{a_{11}}\right)x_3 + \cdots$$
$$+ \left(a_{2n} - a_{1n}\frac{a_{21}}{a_{11}}\right)x_n = b_2 - b_1\frac{a_{21}}{a_{11}} \quad (2)$$

which can be written as

$$a'_{22}x_2 + a'_{23}x_3 + \cdots a'_{2n}x_n = b'_2$$

where $a'_{22} = \left(a_{22} - a_{12}\frac{a_{21}}{a_{11}}\right)$, $a'_{23} = a_{23} - a_{13}\frac{a_{21}}{a_{11}}$)

$\cdots$ etc.

Thus the primes indicate that the orginal element has changed its value.

Similarly, we can multiply the first equation by $-a_{31}/a_{11}$ and add it to the third equation of Eqⁿ (1) and we obtain

$$a'_{32}x_2 + a'_{33}x_3 + \cdots \; a'_{3n}x_n = b'_3 \qquad (3)$$

In a similar fashion, we can eliminate x_1 from the remaining equations and after eliminating x_1 from the last equation (n) of Eqⁿ (1); we obtain the system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots \cdots + a_{1n}x_n = b_1 \\ \quad a'_{22}x_2 + a'_{23}x_3 \cdots + a'_{2n}x_n = b'_2 \\ \quad a'_{32}x_2 + a'_{33}x_3 + \cdots \; a'_{3n}x_n = b'_3 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ \quad a'_{n2}x_2 + a'_{n3}x_3 + \cdots \; a'_{nn}x_n = b'_n \end{array}\right\} \quad (4)$$

We next eliminate x_2 from the last $(n-2)$ equations of Eqⁿ (4).

We have multiplied the first row by $(-a_{21}/a_{11})$ i.e we have divided it by a_{11} which is therefore assumed to be non-zero. For this reason, the first equation in the system (4) is called the pivot equation, and a_{11} is called the pivot or pivotal element. The method obviously fails if $a_{11}=0$.

Now, to eliminate x_2 from the third equation of Eq. (4), we multiply the second equation by $(-a'_{32}/a'_{22})$ and add it to the third Equation. Repeating this process with the remaining equations, we obtain the system.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots\cdots + a'_{2n}x_n = b'_2 \\ a''_{33}x_3 + \cdots\cdots + a''_{3n}x_n = b''_3 \\ \cdots \\ a''_{n3}x_3 + \cdots\cdots + a''_{nn}x_n = b''_n \end{array}\right\} \quad (5)$$

In Eq (5), the double primes indicate that the elements have changed twice. It is easily seen that this procedure can be continued to eliminate x_3 from the fourth equation onwards, x_4 from the fifth equation onwards etc, till we finally obtain the upper-triangular form.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \\ a''_{33}x_3 + \quad a''_{3n}x_n = b''_3 \\ \vdots \\ a^{n-1}_{nn}x_n = b_n^{(n-1)} \end{array}\right\} \quad (6)$$

where a^{n-1}_{nn} indicates that the element a_{nn} has changed $(n-1)$ times.

Step-2 - We now have to obtain the required solution from the system Eq. (6). From the last equation of the system (6), we obtain

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}} \quad \text{—— (7)}$$

This is then substituted in the $(n-1)$th eqn. to obtain x_{n-1} and the process is repeated to compute the other unknowns. We have there first computed x_n then $x_{n-1}, x_{n-2}, \ldots x_2, x_1$, in that order. The process is called back substitution.