Fundamental Theorem of Arithmetic

Every positive integer $n > 1$ is either a prime or a product of primes; this representation is unique, apart from the order in which the factors occur.

Proof:- Either n is a prime or it is composite; in the former case, there is nothing more to prove. If n is composite, then there exists an integer d satisfying $d \mid n$ and $1 < d < n$. Among all such integers d, we choose p_1 to be the smallest (this is possible by the well-ordering principle). Then p_1 must be prime number otherwise it too would have a divisor q with $1 < q < p_1$; but then $q \mid p_1$ and $p_1 \mid n$ imply that $q \mid n$, which contradicts the choice of p_1 as the smallest positive divisor, not equal to 1, of n.

We therefore may write $n = p_1 n_1$, where p_1 is prime and $1 < n_1 < n$. If n_1 happens to be a prime, then we have our representation. In the contrary case, the argument is repeated to produce a second prime number p_2, such that $n_1 = p_2 n_2$; that is

$$n = p_1 p_2 n_2 \qquad 1 < n_2 < n_1$$

If n_2 is a prime, then it is not necessary to go further. Otherwise, we write $n_2 = p_3 n_3$, with p_3 a prime.

$$n = p_1 p_2 p_3 n_3 \qquad 1 < n_3 < n_2.$$

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The decreasing sequence

$$n_0 > n_1 > n_2 > \cdots > 1$$

cannot continue indefinitely, so that after a finite number of steps n_{k-1} is prime. Call it, p_k. This leads to the prime factorization

$$n = p_1 p_2 \cdots p_k$$

To establish the second part of the proof — the uniqueness of the prime factorization — let us suppose that the integer n can be represented as a product of primes in two ways; say

$$n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s, \quad r \leq s$$

where the p_i and q_j are all primes written in increasing magnitude so that

$$p_1 \leq p_2 \leq \cdots \leq p_r \qquad q_1 \leq q_2 \leq \cdots \leq q_s$$

Because $p_1 \mid q_1 q_2 \cdots q_s$ tells us that $p_1 = q_k$ for some k; but then $p_1 \geq q_1$. Similar reasoning gives $q_1 \geq p_1$, whence $p_1 = q_1$. We may cancel this common factor and obtain

$$p_2 p_3 \cdots p_r = q_2 q_3 \cdots q_s.$$

Now repeat the process to get $p_2 = q_2$ and in turn

$$p_3 p_4 \cdots p_r = q_3 q_4 \cdots q_s$$

Continue in this fashion. If the inequalities $r < s$ were to hold, we would eventually arrive at

$$1 = q_{r+1} q_{r+2} \cdots q_s,$$

which is absurd, because each $q_j > 1$.
Hence $r = s$ and

$$p_1 = q_1 \cdot \quad p_2 = q_2, \cdots, p_r = q_r$$

making the two factorizations of n identical.
The proof is now complete.

Of Course, several of the primes that appear in the factorization of a given positive integer may be repeated, as is the case with $360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$.
By collecting like primes and replacing them by a single factor.