Exp. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol. Given linear diff. Eqn.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

in symbolic form

$$(D^2 - 2D + 1)y = xe^x \sin x$$

Its A.E is $D^2 - 2D + 1 = 0$

$(D-1)^2 = 0$

$D = 1, 1$ (roots are real & equal)

Thus $C.F = (c_1 + c_2 x)e^x$

(ii) To find P.I

$$P.I = \frac{1}{(D-1)^2} e^x . x . \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} x \sin x \qquad \because \frac{1}{f(D)} e^{ax} V = e^{ax} \cdot \frac{1}{f(D+a)} V$$

$$= e^x \frac{1}{D^2} x . \sin x$$

$$= e^x . \frac{1}{D} \int x \sin x \, dx \qquad \because \frac{1}{f(D)} = \frac{1}{D} X = \int X dx$$

$$= e^x . \frac{1}{D} \left[x . (-\cos x) - \int 1 . (-\cos x) dx \right]$$

$$= e^x \frac{1}{D} \left[-x \cos x + \sin x \right]$$

$$= e^x \left[\int (-x \cos x) dx + \int \sin x \, dx \right]$$

$$P.I = e^{x}\left[(-x\sin x - \int 1\cdot\sin x\,dx) - \cos x\right]$$

$$= e^{x}\left[-x\sin x - \cos x - \cos x\right]$$

$$= e^{x}\left[-x\sin x - 2\cos x\right]$$

$$P.I = -e^{x}\left[x\sin x + \cos x\right]$$

Hence, the complete solution is

$$y = C.F + P.I$$

$$y = (c_1 + c_2 x)e^{x} - e^{x}\left[x\sin x + \cos x\right]$$

Exp. Solve the :- $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Sol. Given Eqn in symbolic form

$$(D^2+a^2)y = \sec ax$$

Its A.E is $D^2+a^2 = 0$

$$D^2 = -a^2$$

$$D = \pm ia$$

Thus

$$C.F = c_1\cos ax + c_2\sin ax$$

(ii) To find P.I

$$P.I = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D+ia)(D-ia)}\sec ax$$

$$P.I = \frac{1}{D^2+a^2}\sec ax = \frac{1}{(D+ia)(D-ia)}\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax$$

$$= \frac{1}{2ia}\left[\frac{\sec ax}{D-ia} - \frac{\sec ax}{D+ia}\right]$$

Now

$$\frac{1}{(D-ia)}\sec ax = e^{iax}\int \sec ax \cdot e^{-iax}\,dx$$

$$\because \frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx$$

$$= e^{iax}\int \frac{1}{\cos ax}\left[\cos ax - i\sin ax\right]dx$$

$$= e^{iax}\int\left[1 - i\tan ax\right]dx$$

$$= e^{iax}\left[\int dx - i\int \tan ax\,dx\right]$$

$$\frac{1}{D-ia}\sec ax = e^{iax}\left[x + \frac{i}{a}\log|\sec ax|\right] \quad \text{---(i)}$$

Changing i to $-i$ in the Eqn(i)

$$\frac{1}{D+ia} = e^{-iax}\left(x + \frac{i}{a}\log|\sec ax|\right)$$

$$P.I = \frac{1}{2ia}\left[e^{iax}\left(x - \frac{i}{a}\log \sec ax\right) - e^{-iax}\left(x + \frac{i}{a}\log \sec\right.\right.$$

$$= \frac{1}{2ia}\left[x\left(e^{iax} - e^{-iax}\right) + \frac{i}{a}\log|\sec ax|\left(e^{iax} + e^{-iax}\right)\right.$$

$$= \frac{1}{a}\left[\frac{x\left(e^{iax} - e^{-iax}\right)}{2i} - \frac{i}{a}\log|\sec ax|\frac{\left(e^{iax} + e^{-iax}\right)}{2i}\right.$$

$$P.I = \frac{1}{a}\left[x\sin ax - \frac{1}{a}\log|\sec ax| \cdot \cos ax\right]$$

Hence, $y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\log|\sec ax| \cdot \cos ax$