

Exp. Show that the set $\mathcal{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathcal{Q}\}$ is a vector space over \mathcal{Q} (rational number) w.r. to the compositions:

$$(i) (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$$

$$(ii) \alpha(a + b\sqrt{2}) = \alpha a + \alpha b\sqrt{2}, \alpha \in \mathcal{Q}$$

Solution - The set $\mathcal{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathcal{Q}\}$ is an abelian group w.r. to the compositions:-

$$(i) (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$$

The additive identity of $(\mathcal{Q}(\sqrt{2}))$ is $0 = 0 + 0\sqrt{2}$

$$(a + b\sqrt{2}) + 0 + 0\sqrt{2} = 0 + 0\sqrt{2} + a + b\sqrt{2} = a + b\sqrt{2} \quad \forall a, b \in \mathcal{Q}$$

and the additive inverse of $a + b\sqrt{2}$ is $-a - b\sqrt{2}$

then

$$a + b\sqrt{2} + (-a - b\sqrt{2}) = 0$$

Hence, $(\mathcal{Q}(\sqrt{2}), +)$ is abelian group.

Now, properties of scalar multiplication.

Let $\alpha, \beta \in \mathcal{Q}$ and $u = a + b\sqrt{2}$, $v = c + d\sqrt{2}$, $\forall u, v \in \mathcal{Q}(\sqrt{2})$

From (ii) $\alpha(a + b\sqrt{2}) = \alpha a + \alpha b\sqrt{2} \in \mathcal{Q}(\sqrt{2})$

$$1. \alpha(u+v) = \alpha((a+b\sqrt{2}) + (c+d\sqrt{2}))$$

$$= \alpha((a+c) + (b+d)\sqrt{2})$$

$$= \alpha(a+c) + \alpha(b+d)\sqrt{2}$$

$$= \alpha a + \alpha c + (\alpha b + \alpha d)\sqrt{2}$$

$$= \alpha(a + b\sqrt{2}) + \alpha(c + d\sqrt{2})$$

$$= \alpha u + \alpha v$$

$$\therefore \alpha(u+v) = \alpha u + \alpha v$$

$$2. (\alpha + \beta)u = (\alpha + \beta)(a + b\sqrt{2})$$

$$= (\alpha + \beta)a + (\alpha + \beta)b\sqrt{2}$$

$$= \alpha a + \beta a + (\alpha b + \beta b)\sqrt{2}$$

$$= \alpha(a + b\sqrt{2}) + \beta(a + b\sqrt{2})$$

$$= \alpha u + \beta u$$

$$\therefore (\alpha + \beta)u = \alpha u + \beta u$$

$$\begin{aligned}
 3. \quad \alpha(\beta u) &= \alpha(\beta a + \beta b\sqrt{2}) \\
 &= (\alpha\beta)a + (\alpha\beta)b\sqrt{2} \\
 &= (\alpha\beta)(a + b\sqrt{2}) \\
 &= (\alpha\beta)u
 \end{aligned}$$

$$\therefore \alpha(\beta u) = (\alpha\beta)u$$

$$4. \quad 1 \cdot u = 1(a + b\sqrt{2}) = a + b\sqrt{2} = u \quad \forall u \in \mathbb{Q}\sqrt{2}$$

Hence, $\mathbb{Q}\sqrt{2}$ is a vector space over \mathbb{Q} .

Exp. The set $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ is a vector space over \mathbb{R} (real number) w.r.t. matrix addition and multiplication of a matrix by a scalar (α) ; $\alpha \in \mathbb{R}$

Solution We know M_2 is a abelian group w.r.t. matrix addition.

$$\text{Further } \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \in M_2, \alpha \in \mathbb{R}$$

$$\text{Let } \alpha, \beta \in \mathbb{R} \text{ and } A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$\in M_2$. Then $\alpha A \in M_2$ and

$$(i) \quad \alpha(A+B) = \alpha A + \alpha B$$

$$A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$$(ii) \quad (\alpha+\beta)A = \alpha A + \beta A$$

$$(iii) \quad \alpha(\beta A) = (\alpha\beta)A$$

$$\alpha(A+B) = \begin{bmatrix} \alpha(a_1+a_2) & \alpha(b_1+b_2) \\ \alpha(c_1+c_2) & \alpha(d_1+d_2) \end{bmatrix}$$

$$(iv) \quad 1A = A$$

Hence, M_2 is a vector space over \mathbb{R} .

Exp. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$\text{and } \alpha(a_1, a_2) = (\alpha a_1, a_2) \quad \because \alpha \in \mathbb{R}$$

Solution Let $\alpha, \beta \in \mathbb{R}$ and $v = (a_1, a_2) \in V$

$$\text{Then } (\alpha + \beta)v = (\alpha + \beta)(a_1, a_2)$$

$$= ((\alpha + \beta)a_1, a_2)$$

$$= (\alpha a_1 + \beta a_1, a_2)$$

$$= (\alpha a_1, a_2) + (\beta a_1, a_2)$$

$$\therefore (\alpha + \beta)v \neq \alpha v + \beta v$$

Hence V is not a vector space over \mathbb{R}
