Clairaut's form of Differential Equation

Exp. Obtain the singular solution of the differential equation.

$$y = px + \frac{a}{p} \quad \text{where } p = \frac{dy}{dx}$$

Solution We have

$$y = px + \frac{a}{p} \quad \text{——— (1)}$$

Differentiating w.r. to x, we get

$$\frac{dy}{dx} = p \cdot 1 + x \cdot \frac{dp}{dx} - \frac{a}{p^2} \cdot \frac{dp}{dx}$$

$$p = p + \left(x - \frac{a}{p^2}\right)\frac{dp}{dx}$$

$$\Rightarrow \left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad \text{or} \quad x - \frac{a}{p^2} = 0$$

when $\frac{dp}{dx} = 0$

$$dp = 0$$

Integrating $p = c$

Substituting in Eqn (1) we get

$$y = cx + \frac{a}{c} \quad \text{This is the solution}$$

And when $x - \frac{a}{p^2} = 0$ i.e $x = \frac{a}{p^2}$

$$p^2 = \frac{a}{x} \Rightarrow p = \sqrt{\frac{a}{x}}$$

Putting in Eqn (1) then $y = x \cdot \sqrt{\frac{a}{x}} + \frac{a}{\sqrt{a}} \cdot \sqrt{x}$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ Ans

Exp Solve $y = px + \sqrt{a^2p^2+b^2}$ and $p = \frac{dy}{dx}$

Solution We have

$$y = px + \sqrt{a^2p^2+b^2} \quad \text{---- (1)}$$

on differentiating w.r.to x, we get

$$\frac{dy}{dx} = p.1 + x.\frac{dp}{dx} + \frac{1}{2}.(a^2p^2+b^2)^{-1/2}.2p\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{a^2p^2+b^2}}.a^2p\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left(x + \frac{a^2p}{\sqrt{a^2p^2+b^2}}\right) = 0$$

Either $\frac{dp}{dx} = 0$

$$dp = 0 \Rightarrow \int dp = c$$

$$p = c$$

Putting the value of c in Eqn (1) then we get

$$y = cx + \sqrt{a^2c^2+b^2},$$ this is general solution.

Also, when

$$x + \frac{a^2 p}{\sqrt{a^2 p^2 + b^2}} = 0$$

i.e.

$$x\sqrt{a^2 p^2 + b^2} = -a^2 p \quad \text{---} (4)$$

on squaring both sides

$$x^2(a^2 p^2 + b^2) = a^4 p^2$$

$$\Rightarrow a^2 p^2 (a^2 - x^2) = b^2 x^2$$

$$p^2 = \frac{b^2 x^2}{a^2(a^2 - x^2)} \quad \text{---} (3)$$

From Eq (1) & (2)

$$y = px - \frac{a^2 p}{x}$$

$$y = \frac{p(x^2 - a^2)}{x} \quad \text{---} (4)$$

on squaring Eq (4) then

$$y^2 = \frac{p^2 (x^2 - a^2)^2}{x^2} \quad \text{---} (5)$$

From (3) & (5)

$$y^2 = \frac{b^2 (x^2 - a^2)^2}{a^2 (a^2 - x^2)}$$

$$y^2 = -\frac{b^2}{a^2}(x^2 - a^2)$$

$$\frac{y^2}{b^2} = -\frac{x^2}{a^2} + 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is singular solution