Exp. Discuss the convergence of series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n, \quad (x>0)$$

Solution: We have

$$a_n = \frac{(2n)!}{(n!)^2} x^n$$

$$a_{n+1} = \frac{(2n+2)!}{[(n+1)!]^2} x^{n+1}$$

$$\therefore \quad \frac{a_n}{a_{n+1}} = \frac{(2n)!}{(n!)^2} \times \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{x^n}{x^{n+1}}$$

$$= \frac{\cancel{(2n)!}\ (n+1)^2 \cdot \cancel{(n!)^2}}{\cancel{(n!)^2}\,(2n+2)\cdot \cancel{(2n+1)!}} \cdot \frac{1}{x}$$

$$= \frac{(n+1)^2 \cdot \cancel{(2n)!}}{(2n+2)\cdot(2n+1)\,\cancel{(2n)!}} \cdot \frac{1}{x}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)^2}{(2n+2)(2n+1)} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{x} \cdot \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{1}{x} \lim_{n\to\infty} \frac{\cancel{n^2}(1+1/n)^2}{\cancel{n^2}(2+2/n)(2+1/n)}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{x} \cdot \frac{1}{4} = \frac{1}{4x}$$

By Ratio test $\sum a_n$ converges if

$\frac{1}{4x} > 1$ i.e. $x < \frac{1}{4}$ and diverges

if $x > \frac{1}{4}$. The test fails for $x=1$

for $x = 1/4$

$$\frac{a_n}{a_{n+1}} = \frac{1}{1/4} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{4(n+1)^2}{(2n+2)(2n+1)}$$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left[\frac{4(n+1)^2}{(2n+2)(2n+1)} - 1\right]$$

$$= n\left[\frac{\cancel{4n^2} + 4 + 8n - \cancel{4n^2} - 6n - \cancel{2}}{(2n+2)(2n+1)}\right]$$

$$= n\left[\frac{2n+2}{(2n+2)(2n+1)}\right]$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = \frac{\cancel{n}\cdot\cancel{n}(2+3/n)}{\cancel{n}(2+3/n)\cancel{n}(2+1/n)}$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] = \lim_{n\to\infty}\frac{(2+3/n)}{(2+3/n)(2+1/n)}$$

$$= \frac{1}{2} < 1$$

By Raabe's Test Σa_n diverges if $x = \frac{1}{4}$.

Hence Σa_n converges if $x < \frac{1}{4}$ and digerges if $x \geq \frac{1}{4}$.

Exercise (1) Discuss the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{1.3.5 \cdots (4n-5)(4n-3)}{2.4.6 \cdots (4n-4)(4n-2)} \cdot \frac{x^{2n}}{4n},$$

of $x > 0$

∵ Hint $a_n = \frac{1.3.5 \cdots (4n-3)}{2.4.6 \cdots (4n-2)} \cdot \frac{x^{4n}}{4n}$

$$a_{n+1} = \frac{1.3.5 \cdots (4n-3)(4n+1)}{2.4.6 \cdots (4n-2)(4n+2)} \cdot \frac{x^{4n+4}}{(4n+4)}$$