T.D.C. part-II

Electrodynamics

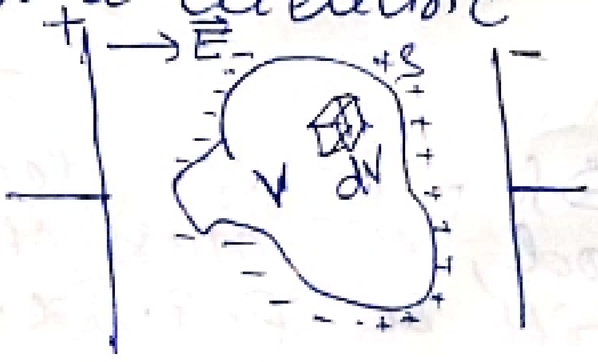
Maxwell's Equation

When charges are in motion, the electric and magnetic fields will be associated with this motion which will have space and time variation. The behaviour of time dependent fields (electric & magnetic) are given by a set of four equations, known as Maxwell's equations of electromagnetic fields. These are

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho \quad \cdots\cdots (a) \\ \vec{\nabla}\cdot\vec{B} &= 0 \quad \cdots\cdots (b) \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad \cdots (c) \\ \vec{\nabla}\times\vec{H} &= \vec{J}+\frac{\partial \vec{D}}{\partial t} \quad \cdots (d) \end{aligned}\right\} \longrightarrow (1)$$

Derivation of Maxwell's EQs.

First Equation: — Consider a surface S bounding a volume V in a dielectric medium as shown in fig (1). In a dielectric medium, the total charge must include both the free and the polarisation charge. Therefore the total charge density at a point in a small volume element dV would be $(\rho+\rho_p)$. Where ρ is the free charge density & ρ_p the charge density due to polarisation



∴ According to Gauss' theorem of electrostatic

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{1}{\epsilon_0}\cdot Q = \frac{1}{\epsilon_0}\oint_V (\rho + \rho_p)dV$$

or, $\oint \vec{E}\cdot\vec{ds} = \frac{1}{\epsilon_0}\oint_V (\rho - \vec{\nabla}\cdot\vec{P})dV$ as $\rho_p = -\vec{\nabla}\cdot\vec{P}$

or, $\oint (\epsilon_0\vec{E})\cdot\vec{ds} = \oint_V \rho dV - \oint_V \vec{\nabla}\cdot\vec{P}dV$

or, $\oint_V (\vec{\nabla}\cdot\epsilon_0\vec{E})dV + \oint(\vec{\nabla}\cdot\vec{P})dV = \oint \rho dV$

(As surface integral is converted into vol^m)

or, $\oint_V \vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P})dV = \oint \rho dV$

or, $\oint_V (\vec{\nabla}\cdot\vec{D})dV = \oint \rho dV$ (∵ Electric displacement $\vec{D} = \epsilon_0\vec{E} + \vec{P}$)

$$\vec{\nabla}\cdot\vec{D} = \rho$$

Thus ~~first law of~~ Maxwell's first Equation is established.

<u>2nd Equation</u>: As the no. of magnetic lines of force entering any arbitrary close surface is exactly the same as leaving it i.e the flux of magnetic induction B across any closed surface is always zero. i.e

$$\int \vec{B}\cdot\vec{ds} = 0$$

Transforming the surface integral into volume integral, we get

$$\int \vec{\nabla}\cdot\vec{B}\,dV = 0.$$

The integrand should vanish for the surface boundary as the volume is arbitrary. ∴ $\vec{\nabla}\cdot\vec{B} = 0$ <u>Proved</u>

Third Equation

According to Faraday law of electromagnetic induction the e.m.f induced in a closed loop is given by

$$e = -\frac{\partial \phi}{\partial t} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$\because$ the flux $\phi = \int_S \vec{B} \cdot d\vec{S}$ where S is any surface.

The e.m.f e is also defined as work done in carrying a unit charge completely around the loop. Thus

$$e = \oint \vec{E} \cdot d\vec{l}$$

$\vec{E}$ is the intensity of the field associated with induced e.m.f

$\therefore$ On equating above equation, we have

$$\oint \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot dS$$

or, $$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$ (using Stoke theorem)

As S is arbitrary, Therefore we have

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is Maxwell's third Equation of electromagnetic field.

~~Both~~ ...... ! -

According to Ampere's law in the circuital form, the work done in carrying a unit magnetic pole once around a closed arbitrary path linked with the current is expressed as

$$\oint \vec{H}\cdot \vec{dl} = I = \int \vec{J}\cdot \vec{ds}$$

where the integral on the right is taken over the surface through which the charge flow corresponding to the current I takes place.

On changing the line integral into surface integral by Stoke's theorem

$$\int_S \text{curl}\,\vec{H}\cdot \vec{ds} = \int \vec{J}\cdot \vec{ds}$$

$$\therefore \text{Curl}\,\vec{H} = \vec{J} \longrightarrow (2)$$

Taking div. on both sides, so we have.

$$\text{div. Curl}\,\vec{H} = \vec{\nabla}\cdot\vec{J}$$

or, $\vec{\nabla}\cdot\vec{J} = 0$ (as div. of curl of a vector is zero) $\longrightarrow (3)$

Eq^n (3) contradicts the equation of continuity i.e $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$. Therefore Maxwell realised that the definition of the total current density is incomplete and suggested to add something to $\vec{J}$.

i.e $\text{Curl}\,\vec{H} = (\vec{J} + \vec{J}') \longrightarrow (4)$

Taking ~~the~~ divergent on both sides, we have

$$\vec{\nabla}\cdot(\vec{J} + \vec{J}') = \text{div Curl}\,\vec{H} = 0$$

or $\vec{\nabla}\cdot\vec{J}' = -\vec{\nabla}\cdot\vec{J} = \frac{\partial \rho}{\partial t}$

or $\vec{\nabla}\cdot\vec{J}' = +\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})$ (from Maxwells first Eq^n)

or $\vec{\nabla}\cdot\vec{J}' = +\vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}$

or $\vec{J}' = \frac{\partial \vec{D}}{\partial t}$

$\therefore$ Eq^n (4) becomes,

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

This is the Maxwell's fourth equation of electromagnetic field. From ~~these~~ equation we conclude that the source of electric field is the changing magnetic field and vice-versa. Also, both the electric field and magnetic field is curly or circular in nature & the their field directions are perpendicular to each other.