* Equations reducible to homogeneous form

The equations of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2x + c_2}$ —— (1)

Case-I ÷ When $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Putting $x = u+h \Rightarrow dx = du$

$y = v+k \Rightarrow dy = dv$

h, k being constants

Then Eq^n (1) becomes

$$\frac{dv}{du} = \frac{a_1u + b_1v + (a_1h + b_1k + c_1)}{a_2u + b_2v + (a_2h + b_2k + c_2)} \quad —— (2)$$

Choose h, k, so that Eq^n (2) may become homogeneous

Thus $\frac{h}{b_1c_2 - b_2c_1} = \frac{k}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - b_1a_2}$

$$\Rightarrow h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - b_1a_2} \qquad \begin{vmatrix} h & k & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$$

$$k = \frac{c_1a_2 - c_2a_1}{a_1b_2 - b_1a_2}$$

When $a_1b_2 - b_1a_2 \neq 0$ then Eq^n (2) becomes

$\frac{dv}{du} = \frac{a_1u + b_1v}{a_2u + b_2v}$ which is homogeneous in u and v and can be solved by putting $v = tu$

Case-II, when $\frac{a_1}{a_2} = \frac{b}{b_2}$ i.e $a_1 b_2 - b_1 a_2 = 0$

then the method fails as h and k become infinite or indeterminate

Now $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{m}$ (say)

$a_2 = m a_1$

$b_2 = m b_1$ and Eqn (1) becomes

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{m(a_1 x + b_1 y) + c_2} \quad \text{--- (4)}$$

Put $a_1 x + b_1 y = t$, so that

$$a_1 + b_1 \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b_1}\left(\frac{dt}{dx} - a_1\right)$$

$\therefore$ Eqn (4) becomes

$$\frac{1}{b_1}\left(\frac{dt}{dx} - a_1\right) = \frac{t + c_1}{mt + c_2}$$

or $\frac{dt}{dx} = a_1 + \frac{b_1 t + b_1 c_1}{mt + c_2} = \frac{(a_1 m + b_1)t + a_1 c_2 + b_1 c_1}{mt + c_2}$

So that the variables are separable. In this solution putting $t = a_1 x + b_1 y$, we get the required solution of Eqn (1).

Exp. Solve $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ —— (i)

Given Eq^n compare with

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

$\Rightarrow a_1 = 1,\ b_1 = 1,\ c_1 = -2$

$a_2 = 1,\ b_2 = -1,\ c_2 = -4$

Thus $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ as Case-I

Putting $x = u+h$, $y = v+k$ (h, k being constants)

So that $\frac{dv}{du} = \frac{v+u+(k+h-2)}{v-u+(k-h-4)}$ —— (ii)

Put $k+h-2 = 0$, $k-h-4 = 0$ $\Rightarrow k = 3$ and $h = -1$

Then Eq^n (ii) becomes

$$\frac{dv}{du} = \frac{v+u}{v-u} \text{ —— (iii)}$$

which is homogeneous in u and v.

Put $v = tu$ differentiate w.r.to u.

$$\frac{dv}{du} = t + u\frac{dt}{du}$$

From (iii) $t + u\frac{dt}{du} = \frac{tu+u}{tu-u} = \frac{t+1}{t-1}$

from Eqⁿ (3)

$$u\frac{dt}{du} = \frac{t+1}{t-1} - t = \frac{1+2t-t^2}{t-1}$$

$$\frac{t-1}{1+2t-t^2}\,dt = \frac{du}{u}$$

Integrating both sides

$$\int \frac{t-1}{1+2t-t^2}\,dt = \int \frac{du}{u} + c$$

Putting $1+2t-t^2 = m$

$$2-2t = dm/dt$$

$$2(1-t)\,dt = dm$$

$$-2(1-t)\,dt = dm$$

$$-\frac{1}{2}\log|m| = \log|u| + c$$

$$-\frac{1}{2}\log|1+2t-t^2| = \log|u| + c$$

$$-\frac{1}{2}\log\left(1+\frac{2v}{u}-\frac{v^2}{u^2}\right) = \log|u| + c$$

$$\log\left|\left(1+\frac{2v}{u}-\frac{v^2}{u^2}\right)\right| = -2\log|u| - 2c$$

$$\log\left|\left(1+\frac{2v}{u}-\frac{v^2}{u^2}\right)\right| + \log u^2 = \log e^{-x}$$

$$\log\left(\frac{u^2+2uv-v^2}{u^2}\right) + \log u^2 = \log e^{-x}$$

$$\log(u^2+2uv-v^2) = \log e^{-2c}$$

$$\Rightarrow u^2+2vu-v^2 = e^{-2c} = c'$$

(iv)

Putting $u = x - h$ & $h = -1 \Rightarrow u = x+1$

$v = y - k$ & $k = 3 \Rightarrow v = y-3$

in Eqn (iv)

$(x+1)^2 + 2(x+1)(y-3) - (y-3)^2 = c'$

$x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$

This is required solution.

Exp. Solve

$$(2x+3y-5)\frac{dy}{dx} + 2x+3y-1 = 0$$

Sol. Given Eqn.

$$\frac{dy}{dx} = \frac{-(2x+3y-1)}{2x+3y-5} \quad \text{——— (i)}$$

where $a_1 = -2$, $b_1 = -3$, $c_1 = 1$

$a_2 = 2$ $b_2 = 3$ $c_2 = -5$

Here $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ i.e $\frac{-2}{2} = \frac{-3}{3} = 1$

Putting $2x+3y = v$ in Eq (i)

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{-v+1}{v-5}$

$\frac{dv}{dx} - 2 = \frac{-3v+3}{v-5} \Rightarrow$ ——— (ii)

From Eqⁿ (ii)

$$\frac{dv}{dx} = 2 + \frac{3-3v}{v-5} = \frac{-v-7}{v-5}$$

$$-\frac{v-5}{(v+7)}\,dv = dx$$

$$dx + \frac{v-5}{v+7}\,dv = 0$$

$$dx + \left(1 - \frac{12}{v+7}\right)dv = 0$$

$$dx + dv - \frac{12}{v+7}\,dv = 0$$

Integrating

$$\int dx + \int dv - 12\int \frac{dv}{v+7} + c = 0$$

$$x + v - 12\log(v+7) = c$$

$$x + (2x+3y) - 12\log(2x+3y+7) = c$$

$$3x + 3y - 12\log(2x+3y+7) = c$$

$$x + y - 4\log(2x+3y+7) = c/3$$

$$x + y - 4\log(2x+3y+7) = c'$$