Abstract Algebra

DEGREE OF A FIELD EXTENSION

Definition :- The dimension of a vector space K over F is called degree of K(F) and this degree of K(F) is always denoted by $[K:F]$.

Example

$Q[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Q\}$ is a field extension of Q. The subset $\{1, \sqrt{2}\}$ forms a basis of $Q[\sqrt{2}]$ over Q. Hence $[Q[\sqrt{2}] : Q] = 2$

Definition :- K is said to be of finite or infinite extension of F according as the degree of K(F) is finite or infinite.

Thus in the above example $Q[\sqrt{2}]$ is a finite extension of Q.

Theorem :-

If K is finite field extension of F and L is a finite field extension of K then prove L is a finite field extension of F and $[L:F] = [L:F][K:F]$

OR

State and prove transivity of finite extension.

Proof :- Suppose the degree (or dimension) of L regarded as vector space over K is m and that of K over F is n. let $\alpha_1, \alpha_2, \ldots, \alpha_m \in L$ and forms a basis for L(K) and let $\beta_1, \beta_2, \ldots, \beta_n \in K$ and forms a basis for K(F). Since $K \subseteq L$ therefore all β_j's $\in L$. Thus the number of elements mn of the set $[\alpha_i \beta_j]$ will form a basis for L(F) for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$.

That is $[L:F] = mn$ and also
$$[L:F] = [L:K]\,[K:F]$$
First, we shall prove that the set $[\alpha_i \cdot \beta_j :$ $1 \le i \le m$, $1 \le i \le m$, $1 \le j \le n$, $m, n \in N\}$ is linearly independent.

For this, we have
$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} (\alpha_i \beta_j) = 0, \; a_{ij} \in F$$
$$\Rightarrow \sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_{ij} \beta_j \right] \alpha_i = 0$$
since $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a basis of $L(K)$ and each $a_{ij} \beta_j \in K$, then, we have
$$\sum_{j=1}^{n} a_{ij} \beta_j = 0.$$
Further, since $\{\beta_1, \beta_2, \ldots, \beta_m\}$ forms a basis for L over K. Then γ can be expressed as the linear Combination of $\alpha_1, \alpha_2, \ldots, \alpha_m$. We have
$$\gamma = \sum_{i=1}^{m} K_i \alpha_j, \; K_i \in K. \quad \text{—— (1)}$$
Further, since $\{\beta_1, \beta_2, \ldots, \beta_n\}$ forms a basis for K over F so it spans K. That is, each element of K can be expressed as the linear Combination of $\beta_1, \beta_2, \ldots, \beta_n$. therefore,
$$K_i = \sum_{i=1}^{n} a_{ij} \beta_j, \; a_{ij} \in F$$
Now from (1) and (2), we get
$$\gamma = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \alpha_i \beta_j \; \forall\, i \le i \le m, \; 1 \le 1 \le n$$

Thus $\gamma \in L$ can be expressed as the linear combination of elements of the set $\{\alpha_i, \beta_j\}$.

Thus the set $\{\alpha_i, \beta_j\}$ forms a basis for L over F.

Consequently,

$$[L:F] = mn$$

i.e. $$[L:F] = [L:K]\,[K:F]$$

Hence the complete theorem.

<u>Corollary</u> :- If L is a finite extension of a field F and K is a subfield of L containing F, then $[K:F] \mid [L:F]$.

<u>Proof</u> :- Since L is a finite extension of F, so that $L(F)$ is a finite dimensional vector space. Also K is a subfield of L contain F, so that F is a subfield of K. Therefore $K(F)$ is also finite dimensional because $L(F)$ is finite dimensional. Hence K is a finite extension of F.

Since $F \subseteq K \subseteq L$ and $L(F)$ and $K(F)$ being finite dimensional, it follows that $L(K)$ is also finite dimensional so that L is a finite extension of F. Then by transitivity of finite extension, we have

$$[L:F] = [L:K]\,[K:F] \Rightarrow [K:F] \mid [L:F]$$