① If S & T are two non-empty sets, then mapping from S to T is a rule. $(f: S \to T)$

→ The cartesian product of two sets S and T is defined as $S \times T = \{(a,b) : a \in S,\ b \in T\}$ → product of sets

we say that $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ & $b_1 = b_2$

Binary Composition '$*$' means addition (+) and multiplication (·) of the elements of two sets.

e.g. $a * b$ ($a+b$ and $a \cdot b$) $a \in S$ & $b \in T$.

Definition - A mapping (function) $* : S \times S \to S$ is called a binary composition on the set S, where $S \times S = \{(a,b) : a, b \in S\}$.

The image of an element $(a,b) \in S \times S$ under $*$ binary composition is usually written as $a * b$ (i.e $a+b$ & $a \cdot b$)

Note If $*$ is a binary composition on a set S then $a * b \in S$ for all $a, b \in S$.

Exp. Addition and multiplication are binary composition in the set N of natural numbers $N = \{1, 2, 3, 4, 5, 6, 7 \ldots\}$ since $a+b \in N$ and $a \cdot b \in N$ $\forall a, b \in N$. However, subtraction is not a binary composition in N. since $2 \in N$ and $3 \in N$

$\left.\begin{array}{l}\text{Addition: } 2+3 = 5 \in N \\ \text{multiplication } 2 \cdot 3 = 6 \in N\end{array}\right\}$ binary composition

but subtraction $2 - 3 = -1 \notin N$ ⟹ Not binary composition

Internal Composition - Let V be any set. Then the mapping $f : V \times V \to V$ is said to be internal composition and it also called vector addition.

Exp. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$f(a,b) = a+b$

External Composition - Let V and F be any two non empty set then the mapping $f : V \times F \to V$ is called to be external composition in V over F also called scalar multiplication Ex. $f : V \times F \to V$, $V\alpha$ or $\alpha V \in V$ for all

Group:- A non-empty set G with a binary (+, ·) composition '$*$' is called a group, if the following conditions are satisfied:

(1) Closure law:- $a*b \in G, \ \forall a, b \in G$

(A1)- $a+b \in G \quad \forall a \in G, b \in G$

(M1)- $a \cdot b \in G \quad \forall a \in G, b \in G$

(2) Associative law:- $(a*b)*c = a*(b*c) \ \forall a,b,c \in G$

(A2)- $(a+b)+c = a+(b+c) \quad a,b,c \in G$

(M2)- $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad a,b,c \in G$

(3) Commutative law: $a*b = b*a \quad \forall a,b \in G$

(A3)- $a+b = b+a \quad \forall a,b \in G$

(M3)- $a \cdot b = b \cdot a \quad \forall a,b \in G$

(4) Existence of Identity:- $\exists e \in G : a*e = e*a = a$

(A4) Addition identity $e=0$, $a+0 = 0+a = a \quad \forall a \in G$

(M4) Multiplication identity $e=1$, $a \cdot 1 = 1 \cdot a = a, \ \forall a \in G$

(5) Existence of inverse:- $\forall a \in G, \exists b \in G$

$a*b = b*a = e$

(A5) Addition inverse $(b = -a)$, $a+(-a) = (-a)+a = 0$

(M5) Multiplication inverse $(b = a^{-1})$ $a \cdot a^{-1} = a^{-1} a = 1$

A group $(G, *)$ is called an abelian group if
commutative

$a*b = b*a \quad \forall a,b \in G$

$a+b = b+a \quad \forall a,b \in G$

$a \cdot b = b \cdot a \quad \forall a,b \in G$.

Field – A non-empty set F with ~~two~~ binary compositions $(+, \cdot)$ denoted by $+$ and $\cdot$ is called a field, if the following these property are satisfied.

A 1) $a+b \in F \;\; \forall a, b \in F$ — Closure law

A 2) $a+b = b+a \;\; \forall a, b \in F$ — Commutative law

A 3) $a+(b+c) = (a+b)+c \;\; \forall a, b, c \in F$ — Associative law

A 4) There exists an element $0 \in F$ such that $a+0 = 0+a = a$ st. $\forall a \in F$ (Additive identity)

A 5) $\forall a \in F$, there exists some element $-a$ in F s.t. $-a \in F$ such that $a+(-a) = 0 = (-a)+a$ (additive inverse)

M (6) $a \cdot b \in F \;\; \forall a, b \in F$

M (7) $a \cdot b = b \cdot a \;\; \forall a, b \in F$

M (8) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \;\; \forall a, b, c \in F$

M (9) $\exists\, e \in F$: $a \cdot e = e \cdot a = a \;\; \forall a \in F$, e is called multiplicative identity of F. As e by 1, so that $a \cdot 1 = 1 \cdot a = a \;\; \forall a \in F$

M (10) $\forall a \in F$ & $a \neq 0$, $\exists\, b \neq 0 \in F$ such that $a \cdot b = b \cdot a = 1$

we write b as a^{-1} and all in the multiplicative inverse of a. Thus $a \cdot a^{-1} = a^{-1} \cdot a = 1$

(11) $a \cdot (b+c) = a \cdot b + a \cdot c \;\; \forall a, b, c \in F$ (Distributive law)

Exp. The set Q of all rational numbers is a field w.r.t usual addition and multiplication.

Vector Space (V) over a Field (F), V(F): Let V be a non-empty set and F is a field, such that V is a vector space over F, if the following conditions/properties/axioms are satisfied.

1- $u+v \in V$, $\forall$ $u \in V$ and $v \in V$
2. $(u+v)+w = u+(v+w)$, $\forall$ $u, v, w \in V$
3. $u+v = v+u$, for all $(\forall)$ $u\, v \in V$
4. There exists an element $e=0$ in V, such that $v+0 = 0+v = v$, $\forall$ $v \in V$, $0 \in V$ is called zero vector in V.
5. For each $u \in V$, there exists some $v \in V$ such that $u+v = v+u = 0$. v as $-u$. Thus
$$u+(-u) = (-u)+u = 0$$
$-u$ is called the negative or additive inverse.
6. $\alpha(u+v) = \alpha u + \alpha v$, for all $\alpha \in F$ & $u, v \in V$
7. $(\alpha+\beta)v = \alpha v + \beta v$, $\forall$ $\alpha, \beta \in F$ & $v \in V$
8. $\alpha(\beta v) = (\alpha\beta)v$, $\forall$ $\alpha, \beta \in F$ & $v \in V$
9. If 1 is unity of F, then $1v = v$ for all $v \in V$

* A vector space V over a field F is denoted by V(F). The elements of V are called vectors and the elements of F are called scalars. Vectors will be written as u, v, w, x, y, z and scalars as $\alpha, \beta, \gamma, a, b, c$ etc.

I → The first 5 properties of a vector space imply that $(V, +)$ is an abelian group.

II → The zero elements of V and F are both denoted by 0. $V = \{0, 0, 0, 0, 0\}$
$F = (0, 0, 0, 0, 0)$