LINEAR PROGRAMMING

CONVEX SET AND THEIR PROPERTIES

Theorem VI :- If the Convex set of the feasible solutions of $Ax=b$, $x \geqslant 0$, is a Convex polyhedron, then at least one of the extreme point gives an optimal solution.

Proof :- In the last Corralary we have proved that the extreme points of the Convex set of feasible solutions of $Ax=b$, $x \geqslant 0$ are finite in number.

Let $x_1, x_2, \ldots, x_n$ be the extreme points of the set X of all the feasible solutions of $Ax=b$, $x \geqslant 0$. Let Z be the objective function which is to the maximized be given by

$$Z = cx.$$

If $x^* \in X$ is the optimal solution, then

$$\text{Max. } Z = cx^*.$$

Now if x^* is an extreme point, then the theorem is proved.

Now if x^* is not an extreme point in X. Then since X is Convex polyhedron therefore x^* Can be expressed as a Convex Combination of the extreme points of X.

$$\text{i.e. } x^* = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$$

$$= \sum_{i=1}^{k} \lambda_i \cdot x_i \text{ , } \lambda_i \geqslant 0 \text{ and } \sum \lambda_i = 1.$$

$\therefore\ z^* = cx^* = c(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)$

$= \lambda_1 cx_1 + \lambda_2 cx_2 + \cdots + \lambda_k cx_k$

If maximum of cx_i is cx_p, then

$$Z^* \leq (\lambda_1 + \lambda_2 + \cdots + \lambda_2) \cdot cx_p$$

or, $Z^* \leq cx_p$

cx_p is the maximum value of z.

Therefore

Max. $Z = cx_p$

or, $cx^* = cx_p$

i.e. $x^* = x_p$ (one of the extreme points)

Hence the optimal solution is attained at the extreme point.

which proves the theorem.