* Convergence of the Infinite integral $\int_1^\infty a(x)\,dx$

Def. The infinite integral $\int_1^\infty a(x)\,dx$ is said to be (i) convergent if $\lim_{t\to\infty}\int_1^t a(x)\,dx$ is finite, and (ii) divergent if $\lim_{t\to\infty}\int_1^t a(x)\,dx$ is infinite (∞).

Theorem If $f(x)\geq 0\ \forall\, x\geq 1$, then $\int_1^\infty f(x)\,dx$ is convergent iff there exists a (+ve) number k, such that for all $t\geq 1$, $\int_1^t f(x)\,dx\leq k$.

Exp. Examine the convergence and divergence of (1) $\int_1^\infty \frac{dx}{x}$ & (ii) $\int_1^\infty \frac{dx}{1+x^2}$

Sol. (1) We have.

$$\int_1^t \frac{dx}{x} = |\log x|_1^t = \log t - \log 1$$

$$\int_1^t \frac{dx}{x} = \log t$$

$$\lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}(\log t) = \infty$$

Hence $\int_1^\infty \frac{dx}{x}$ diverges.

(ii) $\int_1^t \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_1^t = \tan^{-1} t - \tan^{-1} 1$

$$\int_1^t \frac{dx}{1+x^2} = \tan^{-1} t - \tan^{-1} \tan \pi/4$$

$$\int_1^t \frac{dx}{1+x^2} = \tan^{-1} t - \pi/4$$

$$\lim_{t\to\infty} \int_1^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \left(\tan^{-1} t - \pi/4\right)$$

$$= \tan^{-1} \infty - \pi/4$$

$$= \tan^{-1} \tan \pi/2 - \pi/4$$

$$\lim_{t\to\infty} \int_1^t \frac{dx}{1+x^2} = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4$$

Hence $\int_1^\infty \frac{dx}{1+x^2}$ is convergence.