Cauchy's n^{th} Root Test :-

Let $\sum a_n$ be a positive term series such that

$$\lim_{n\to\infty} (a_n)^{1/n} = l$$

Then (i) $\sum a_n$ converges if $l<1$

(ii) $\sum a_n$ diverges if $l>1$

Test fails if $l=1$.

<u>Proof</u> Since

$$\lim_{n\to\infty} (a_n)^{1/n} = l \quad \text{——— (1)}$$

Case 1 - let $l<1$

We can choose some $\epsilon>0$ such that $l+\epsilon<1$ or $\alpha<1$, $\alpha = l+\epsilon$.

Using (1) there exists a positive integer m_1

$$\Rightarrow \quad |(a_n)^{1/n} - l| < \epsilon \qquad \forall\ n \geq m_1$$

$$\text{or} \quad l-\epsilon < (a_n)^{1/n} < l+\epsilon \qquad \forall\ n \geq m_1$$

Consider $(a_n)^{1/n} < l+\epsilon = \alpha \qquad \forall\ n\geq m_1$

$$a_n < \alpha^n \qquad \forall\ n \geq m_1 \quad \text{——— (2)}$$

Since $\sum \alpha^n = \alpha + \alpha^2 + \cdots$ being a geometric series with common ratio $\alpha<1$ is convergent So by First Comparison Test in (2), $\sum a_n$ is Convergent.

Case II. let $l > 1$

We can choose another $\epsilon > 0$ such that $l - \epsilon > 1$ or $\beta > 1$, $\beta = l - \epsilon$

Using (1), there exists a positive integer m_2

$\Rightarrow \quad |(a_n)^{1/n} - l| < \epsilon \quad \forall\, n \geq m_2$

or $\quad l - \epsilon < (a_n)^{1/n} < l + \epsilon \quad \forall\, n \geq m_2$

Consider $\quad l - \epsilon < (a_n)^{1/n} \quad \forall\, n \geq m_2$

$\beta < (a_n)^{1/n} \quad \forall\, n \geq m_2$

$\beta^n < a_n \quad \forall\, n \geq m_2$ — (3)

Since $\sum \beta^n$ being a geometri series with common ration $\beta > 1$ is divergent so by first comparison test as applied in (3) $\sum a_n$ is divergent.

Case-III We shall give examples of two series one convergent and the other divergent but both satisfying $\lim_{n\to\infty} (a_n)^{1/n} = 1$

The series $\sum a_n = \sum \frac{1}{n}$ is divergent but $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} (1/n)^{1/n} = 1$

The series $\sum a_n = \sum \frac{1}{n^2}$ is convergent but $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} (\frac{1}{n^2})^{1/n} = 1$

Exp. Test for convergence the series

(i) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Sol. (i) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n} = \sum_{n=2}^{\infty} \left(\frac{1}{\log n}\right)^n$

$$a_n = \left(\frac{1}{\log n}\right)^n$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left\{\left(\frac{1}{\log n}\right)^n\right\}^{1/n}$$

$$\lim_{n\to\infty} \left(\frac{1}{\log n}\right)^n \quad = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^n} = 0 < 1$$

Hence the given series is convergent.

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^n} \Rightarrow \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left\{\left(\frac{1}{n}\right)^n\right\}^{1/n}$

$$= \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

Hence the given series is convergent.