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## Legendre and Associated legendre functions

### Legendre Polynomials :-

Schrodinger equation for rotational motion of a particle is given as

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\cdot\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\phi^2} + \frac{8\pi^2 m r^2}{h^2}E\psi = 0 \quad \text{--- (1)}$$

where $r$ is the radius vector and $(\theta,\phi)$ are angular variables.

The partial differential Schrodinger equation can be solved by separating the variables.

Let us suppose, the function $\psi(\theta,\phi)$ is a product of two function $P(\theta)$ & $F(\phi)$.

i.e. $$\frac{\partial\psi}{\partial\theta} = F(\phi)\cdot\frac{dP}{d\theta}$$

& $$\frac{\partial^2\psi}{\partial\theta^2} = F(\phi)\cdot\frac{d^2P}{d\theta^2}$$

and w.r.t $\phi$ we get —

$$\frac{\partial\psi}{\partial\phi} = P(\theta)\frac{dF}{d\phi}$$

& $$\frac{\partial^2\psi}{\partial\phi^2} = P(\theta)\cdot\frac{d^2F}{d\phi^2}$$

substituting these relations into eq$^n$ (1) we get —

$$F\cdot\frac{d^2P}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\cdot F\cdot\frac{dP}{d\theta} + \frac{1}{\sin^2\theta}\cdot P\cdot\frac{d^2F}{d\phi^2} + \frac{8\pi^2 m r^2}{h^2}EPF = 0$$

On multiplying by $\frac{\sin^2\theta}{PF}$,

$$\frac{\sin^2\theta}{P}\cdot\frac{d^2P}{d\theta^2}+\cos\theta.\sin\theta\cdot\frac{1}{P}\cdot\frac{dP}{d\theta}+\frac{1}{F}\cdot\frac{d^2F}{d\phi^2}+\frac{8\pi^2mr^2}{h^2}\sin^2\theta.E=0$$

or $$\frac{\sin^2\theta}{P}\cdot\frac{d^2P}{d\theta^2}+\cos.\theta.\sin\theta\cdot\frac{1}{P}\cdot\frac{dP}{d\theta}+\frac{8\pi^2mr^2}{h^2}\sin^2\theta.E=-\frac{1}{F}\frac{d^2F}{d\phi^2} \quad \text{—(2)}$$

The eqn on LHS depends only on '$\theta$' and eqn on RHS depends on '$\phi$' only. If $\phi$ is constant and $\theta$ varies, RHS remains constant and LHS will also remain equal to the same constant. Thus equations —(2) splits into the following two ordinary differential equation

$$-\frac{1}{F}\frac{d^2F}{d\phi^2}=M^2 \quad \text{—(3)}$$

and

$$\frac{\sin^2\theta}{P}\cdot\frac{d^2P}{d\theta^2}+\cos.\theta.\sin\theta\cdot\frac{1}{P}\cdot\frac{dP}{d\theta}+\frac{8\pi^2mr^2}{h^2}\sin^2\theta.E=M^2 \quad \text{—(4)}$$

On multiplying eqn —(4) by $P/\sin^2\theta$, we get

$$\frac{d^2P}{d\theta^2}+\frac{\cos.\theta}{\sin\theta}\cdot\frac{dP}{d\theta}+\beta.P=\frac{M^2P}{\sin^2\theta}$$

where, $$\beta=\frac{8\pi^2mr^2}{h^2}.E$$

This eqn is known as Legendre differential equation. Such equation are solved using Polynomial methods. For

- each of value of $|M|$ ( || Modulus sign Considers only the magnitude but not the sign of M). There will be a corresponding Legendre equation and a new set of Solution.

let $|M| = 0$ The legendre equation is —

$$\frac{d^2P}{d\theta^2} + \frac{\cos\theta}{\sin\theta} \cdot \frac{dP}{d\theta} + \beta P = 0$$

The solution to this equation exists as polynomial in $\theta$ called " Legendre Polynomial" given as

$$P_l(x) = \frac{1}{2^l \, l!} \frac{d^l}{dx^l}(x^2-1)^l$$

where $x = \cos\theta$ and $l$ is an integer including zero.

for $|M| \neq 0$, the solutions have the form

$$P_L^{|M|}(x) = (1-x^2)^{|M|/2} \cdot \frac{d^{|M|}}{dx^{|M|}} \cdot P_l(x)$$

These polynomials are called "Associated legendre Polynomials".

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