

Legendre and Associated Legendre functions
Legendre Polynomials :-

Schrodinger equation for rotational motion of a particle is given as

$$\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \cdot \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin^2 \theta} \cdot \frac{\partial^2 \psi}{\partial \phi^2} + \frac{8\pi^2 m r^2}{h^2} E \psi = 0 \quad \text{--- (1)}$$

where r is the radius vector and (θ, ϕ) are angular variables.

The partial differential Schrodinger equation can be solved by separating the variables.

Let us suppose, the function $\psi(\theta, \phi)$ is a product of two function $P(\theta)$ & $F(\phi)$.

$$\text{i.e. } \frac{\partial \psi}{\partial \theta} = F(\phi) \cdot \frac{dP}{d\theta}$$

$$\& \quad \frac{\partial^2 \psi}{\partial \theta^2} = F(\phi) \cdot \frac{d^2 P}{d\theta^2}$$

and w.r.t ϕ we get -

$$\frac{\partial \psi}{\partial \phi} = P(\theta) \frac{dF}{d\phi}$$

$$\& \quad \frac{\partial^2 \psi}{\partial \phi^2} = P(\theta) \cdot \frac{d^2 F}{d\phi^2}$$

substituting these relations into eqⁿ (1) we get -

$$F \cdot \frac{d^2 P}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \cdot F \cdot \frac{dP}{d\theta} + \frac{1}{\sin^2 \theta} \cdot P \cdot \frac{d^2 F}{d\phi^2} + \frac{8\pi^2 m r^2}{h^2} E P F = 0$$

On multiplying by $\frac{\sin^2 \theta}{PF}$,

$$\frac{\sin^2 \theta}{P} \cdot \frac{d^2 P}{d\theta^2} + \cos \theta \cdot \sin \theta \cdot \frac{1}{P} \cdot \frac{dP}{d\theta} + \frac{1}{F} \cdot \frac{d^2 F}{d\phi^2} + \frac{8\pi^2 m r^2}{h^2} \sin^2 \theta \cdot E = 0$$

$$\text{or } \frac{\sin^2 \theta}{P} \cdot \frac{d^2 P}{d\theta^2} + \cos \theta \cdot \sin \theta \cdot \frac{1}{P} \cdot \frac{dP}{d\theta} + \frac{8\pi^2 m r^2}{h^2} \sin^2 \theta \cdot E = -\frac{1}{F} \frac{d^2 F}{d\phi^2} \quad \text{--- (2)}$$

The eqⁿ on LHS depends only on ' θ ' and eqⁿ on RHS depends on ' ϕ ' only. If ϕ is constant and θ varies, RHS remains constant and LHS will also remain equal to the same constant. Thus equations --- (2) splits into the following two ordinary differential equation

$$-\frac{1}{F} \frac{d^2 F}{d\phi^2} = M^2 \quad \text{--- (3)}$$

and

$$\frac{\sin^2 \theta}{P} \cdot \frac{d^2 P}{d\theta^2} + \cos \theta \cdot \sin \theta \cdot \frac{1}{P} \cdot \frac{dP}{d\theta} + \frac{8\pi^2 m r^2}{h^2} \sin^2 \theta \cdot E = M^2 \quad \text{--- (4)}$$

on multiplying eq^s --- (4) by $\frac{P}{\sin^2 \theta}$, we get

$$\frac{d^2 P}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \cdot \frac{dP}{d\theta} + \beta \cdot P = \frac{M^2 P}{\sin^2 \theta}$$

where, $\beta = \frac{8\pi^2 m r^2}{h^2} \cdot E$

This eqⁿ is known as Legendre differential equation. Such equation are solved using Polynomial methods. For

- each of value of $|M|$ (11) modulus sign considers only the magnitude but not the sign of M . There will be a corresponding Legendre equation and a new set of solution.

Let $|M| = 0$ The Legendre equation is -

$$\frac{d^2 P}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \cdot \frac{dP}{d\theta} + \beta P = 0$$

The solution to this equation exists as polynomial in θ called "Legendre Polynomial" given as

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

where $x = \cos \theta$ and l is an integer including zero.

for $|M| \neq 0$, the solutions have the form

$$P_L^{(M)}(x) = (1-x^2)^{|M|/2} \cdot \frac{d^{|M|}}{dx^{|M|}} \cdot P_l(x)$$

These polynomials are called "Associated Legendre Polynomials".

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