Unit - II

* Quantum Mechanical treatment of linear Harmonic Oscillator :-

Since the potential energy of oscillation at any point 'x' is —

$$2\pi^2 \nu_0^2 \mu x^2$$

The Hamiltonian operator for a harmonic oscillator is given by —

$$\hat{H} = \frac{-h^2}{8\pi^2\mu}\frac{d^2}{dx^2} + 2\pi^2\nu_0^2 \mu x^2$$

The schrödinger equation $\hat{H}\psi = E\psi$ for such a motion may be written as —

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2\mu}{h^2}\left(E - 2\pi^2\nu_0^2 \mu x^2\right)\psi = 0 \quad \text{— (4)}$$

Let,

$$\alpha = \frac{8\pi^2 mE}{h^2} \quad \& \quad \beta = \frac{4\pi^2\nu_0 . \mu}{h} \quad \text{— (5)}$$

So, Eqs — (4) becomes —

$$\frac{d^2\psi}{dx^2} + (\alpha - \beta x^2)\psi = 0 \quad \text{— (6)}$$

Let, $y = \sqrt{\beta}\, x$

So, $dx = \frac{dy}{\sqrt{\beta}}$ & $\frac{d^2\psi}{dx^2} = \beta\frac{d^2\psi}{dy^2}$

Substituting these values in eqs — (6) we get,

$$\beta . \frac{d^2\psi}{dy^2} + (\alpha - \beta y^2)\psi = 0$$

or, $$\frac{d^2\psi}{dy^2} + \left(\frac{\alpha}{\beta} - y^2\right)\psi = 0 \qquad \text{---} (7)$$

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* Solution of this Equation :-

let $y \gg \frac{\alpha}{\beta}$.

then eqⁿ - (7) reduces to

$$\frac{d^2\psi}{dy^2} - y^2\psi = 0$$

The solution of this equation may be as $\psi = \exp\left(\pm y^2/2\right)$

But the solution $\psi = \exp\left(\pm y^2/2\right)$ is not acceptable because in that case ψ will be rapidly increase tending to ∞ as y approaches $\pm\infty$. So, the other solution is

$$\psi = \exp\left(-y^2/2\right)$$

So, the general solution for eqⁿ - (7) may be written as

$$\psi = F(y)\exp\left(-y^2/2\right) \qquad \text{---} (8)$$

Where, $F(y)$ or F is same function of y whose form has to be determined by finding the differential eqⁿ for it. differentiating eqⁿ - (8) w.r.t y, we obtain,

$$\frac{d^2\psi}{dy^2} = \left\{\frac{d^2F}{dy^2} - 2y\frac{dF}{dy} + \left(y^2 - 1\right)F\right\}\exp\left(-y^2/2\right)$$

--- (9)

Hence, by substituting for ψ and $\frac{d^2\psi}{dy^2}$ from eqn (8) & (9) respectively in the eqn – (6), we get

$$\left[\frac{d^2F}{dy^2} - 2y\frac{dF}{dy} + (y^2-1)F\right]\exp\left(-y^2/2\right) + \left[\alpha/\beta - y^2\right]\exp\left(-y^2/2\right)F = 0$$

or, $$\left[\frac{d^2F}{dy^2} - 2y\frac{dF}{dy} + \left(\alpha/\beta - 1\right)F\right]\exp\left(-y^2/2\right) = 0 \qquad (10)$$

Since, $\exp\left(y^2/2\right) \neq 0$ except $y = \pm\infty$

the expression within brackets must be zero.

i.e. $$\frac{d^2F}{dy^2} - 2y\frac{dF}{dy} + \left(\alpha/\beta - 1\right)F = 0 \qquad (11)$$

This eqn – (11) is known as Hermite's differential Equation.

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